Solutions for DE topic test 2009

Pblm #1: Which of the following is a particular solution of $y'' - 4y = 6x - 4x^3$?

Solution: Building a particular solution in the form of cubic polynomial gives

ANSWER: C. x^3

Pblm #2: Which equilibrium points of $N' = N(N-1)(2-N)$ are stable considering only non-negative solutions?

Solution: N increases for $1 < N < 2$ and decreases on $0 < N < 1$ and decreases on $tN > 2$

ANSWER: C. 0 and 2

Pblm #3: Starting at 9am, oil is pumped into a storage tank at a rate of $150t^{\frac{1}{2}} + 25$ (in gal/hr), for time t in hours after 9am. How many gallons will have been pumped into the tank at 1pm?

Solution:

$$\int_0^4 (150t^{\frac{1}{2}} + 25)dt = 100(4)^{3/2} + 25(4)$$

ANSWER: D. 900

Pblm #4: A boat moves away from a dock along a straight line, with an acceleration at time t (in seconds) given by $a(t) = 12t - 4$ (in ft/sec^2). If at time $t = 0$, the boat had a velocity of 8 ft/sec and was 15 feet from the dock, find its distance in feet from the dock at time $t = 2$ seconds.

Solution: $v = 6t^2 - 4t + 8$ using $v(0) = 8$. Then $s = 2t^3 - 2t^2 + 8t + 15$ using $s(0) = 15$. Calculate $s(2)$.

ANSWER: B. 39

Pblm #5: Find $y(0)$ if y solves $y'' + y = 4x + 10\sin(x)$ with $y(\pi) = 0, y'(\pi) = 2$.

Solution: The solution to the homogeneous equation is $y_h(x) = A\cos x + B\sin x$. A particular solution can be found by undetermined coefficients (noting the resonance case for $\sin x$): $y_p(x) = ax + b + x(c\cos x + d\sin x)$. Plugging the form y_p into the ODE gives $ax + b + 2(-c\sin x + d\cos x) = 4x + 10\sin x$, hence $a = 4$, $b = 0$, $c = -5$, $d = 0$.

The general solution for the inhom ODE is $y = A\cos x + B\sin x + 4x - 5x\cos x$. From $y(\pi) = 0$, we get $-A + 4\pi + 5\pi = 0$, so $A = 9\pi$. We don't even need B to find $y(0) = A = 9\pi$.

ANSWER: D

Pblm #6: An 8-pound weight stretches a spring 2 feet beyond its natural length. The weight is then pulled down another .5 feet and released with an initial upward velocity of 6 ft/sec. Find a formula for the position of the mass at any time t, given $y(0) = -1/2$.

Solution: $y'' + 4y/m = 0$ and $m = W/g = 8/32$. Use $y(0) = -1/2$ and $y'(0) = 6$ gives $-\frac{1}{2}\cos 4t + \frac{3}{2}\sin 4t$.

ANSWER: E. none of the above

Pblm #7: Using the general solution of

$$\frac{dy}{dt} + \tan(t)y = \cos(t)$$

on the interval $(-\pi/2, \pi/2)$, calculate the limit of $y(t)$ as t goes to $\pi/2$ from the left.

Solution: An integrating factor is $1/\cos(t)$. With it, the ODE becomes $\frac{d}{dt}(y(t)/\cos t) = 1$. The general solution is $y(t) = (t+C)\cos t$. As $t \to \pi/2-$, $y(t) \to 0$ for any C.

ANSWER: A

Pblm #8: For the solution of $y' - 2xy = x$ with $y(0) = 0$, find $y(1)$.

Solution: An integrating factor is e^{-x^2}. The ODE is equaivalent to $(e^{-x^2}y)' = xe^{-x^2}$. Definite integration starting at $x = 0$ with $y(0) = 0$ gives $e^{-x^2}y - 0 = -\frac{1}{2}e^{-x^2} + \frac{1}{2}$ and therefore $y = \frac{1}{2}(e^{x^2} - 1)$.

We conclude $y(1) = \frac{1}{2}(e - 1)$.

ANSWER: C

Pblm #9: Find an implicit solution of

$$\frac{dy}{dx} = \frac{y\cos x}{1 + 2y^2},\ y(0) = 1.$$

Solution: The equation is separable.

$$\int_1^y \frac{(1+2y^2)\,dy}{y} = \int_0^x \cos x\,dx$$
$$\ln y + y^2 - 1 = \sin x$$

The absolute value $|y|$ in the suggested solutions is superfluous since those y that arise from a solution of the initial value problem will be positive.

ANSWER: D

Pblm #10: Find $y(0)$, if y solves $y''+4y'+4y = 0$ with $y(1) = 0$, $y(2) = e^{-4}$.

Solution: The characteristic eqn is $r^2 + 4r + 4 = 0$, with a double root -2. The general solution is therefore $y = (C_1 + C_2x)e^{-2x}$. $y(1) = 0$ implies $C_2 = -C_1$. $y(2) = e^{-4}$ then means $-C_1e^{-4} = e^{-4}$, i.e., $C_1 = -1$. Now $y(0) = C_1 = -1$.

ANSWER: E

Pblm #11: Which of the following implicitly defines a solution of this equation?

$(1 + 3x \sin y)dx = x^2 \cos y dy$.

This would be exact if $\frac{\partial}{\partial y}(1 + 3x \sin y)$ were equal to $\frac{\partial}{\partial x}(-x^2 \cos y)$. Observing that this is not the case, but that the y dependent terms do match already, we try an integrating factor that depends on x only, let's call it $\mu(x)$. We want to determine μ in such a way that $\frac{\partial}{\partial y}[\mu(x)(1 + 3x \sin y)] = \frac{\partial}{\partial x}[\mu(x)(-x^2 \cos y)]$, i.e., $\mu(x)\,3x = -x^2\mu'(x) - 2x\mu(x)$. This implies $\mu(x) = cx^{-5}$, and with this choice we get the exact ODE $(x^{-5} + 3x^{-4} \sin y)dx - x^{-3} \cos y dy = 0$.

The solution is
$$-\frac{1}{4}x^{-4} - x^{-3} \sin y = C$$

ANSWER: D (The -4 times the C here is the C in the answer.)

Pblm #12: A ball is thrust up vertically from the ground into the air and hits the ground 2.5 seconds later. What is the maximum height of the ball in feet? Assume that air resistance is negligible. (Use acceleration due to gravity -32 ft/sec^2)

Working with $g = -32$, we have $v(t) = v_0 + gt$ (upwards counts as positive) and $h(t) = v_0t + \frac{1}{2}gt^2$. Since $h(2.5) = 0$, we obtain $v_0 = -\frac{1}{2}gt = 40$ and hence the maximum height (obtained at $\frac{1}{2}\times 2.5$) is $h(\frac{5}{4}) = 40\frac{5}{4} - 16(\frac{5}{4})^2 = 25$. The maximum height is 25 ft.

ANSWER: B

Pblm #13: Solve for y, such that $y' = \frac{2xy}{1+x^2}$ and $y(2) = 5$.

Solution: Separable.

$\int_5^y \frac{dy}{y} = \int_2^x \frac{2x}{1+x^2}\,dx$. So $\ln(y/5) = \ln(1+x^2) - \ln 5$, i.e., $y = 1 + x^2$.
ANSWER: E

Pblm #14: At time $t = 0$, a tank contains 10 pounds of salt dissolved in 100 gallons of water. Assume that water containing 0.25 pounds of salt per gallon is entering the tank at a rate of 3 gallons per minute and that a well-stirred solution is leaving the tank at the same rate. Find an expression for the amount of salt in the tank at time t.

Solution: Let y be the amount of salt in pounds in the tank. $y' = 3/4 - 3y/100$. Use IC, $y(0) = 10$, and get $y = 25(1 - e^{-0.03t}) + 10e^{-0.03t}$

E. none of the above

Pblm #15: In a book, you read the following text:
"The initial value problem

$$y^{[\text{redacted}]} + y' + 5y = \frac{\cos^2 t}{e^t + e^{-t}}\ , \ y(0) = 1,\ y'(0) = 2,\ y''(0) = 3,\ y'''(0) = -1$$

has exactly one solution."

Alas a smudge has made the derivative in the first term ("y[redacted]") unreadable. Reconstruct the correct reading.

Solution: By 'rule of thumb', one needs as many initial conditions as the order of the ODE to get exactly one solution. The order of the DE should therefore be 4.

'Rule of thumb' is no guarantee to work always; however there is a variety of hypotheses under which this principle works reliably: (1) linear equations with (at least) continuous coefficients, where the coefficient of the highest order term doesn't vanish at the initial time; and the initial conditions specifying all derivatives of lower order than the order of the ODE. This is the situation here. (2) A more general case is: The ODE can be written in the form 'highest derivative of y' = expression in terms of lower derivatives of y and x, and this expression has partial derivatives with respect to all $y^{(k)}$ and is continuous in x.

When I say 'lower order derivatives', I mean to include the function itself as '0^{th} derivative.

ANSWER: C

Pblm #16: For the solution of the IVP

$$\frac{y'}{2+\cos x} - e^{-y} = 0\,,\ y(0) = 0$$

find $y(\pi/2)$.

Solution: Separable. $\int_0^y e^y\,dy = \int_0^x (2+\cos x)\,dx$.

$e^y - 1 = 2x + \sin x$. For $x = \pi/2$, we find $y = \ln(2+\pi)$.

ANSWER: B

Pblm 17: In 1980, the population of the kingdom of Edenia was 1 million. The 2000 census then found a population of 1.44 million. Assuming Malthusian growth, i.e., the growth rate is proportional to the population size, how many people should have lived in Edenia in 1990?

Solution: With Malthusian growth, we have the population size $P(t) = P_0 e^{at}$ with some constant a that is not given in the problem and depends on birth and death rate in the population. Whichever these parameters are, we have $P(10) = \sqrt{P(0)P(20)}$ (starting the clock in 1980). $\sqrt{1.44} = 1.20$.

ANSWER: B

Pblm 18: Which of the following is an integrating factor for the ODE $yy'y'' = 1$?

Solution: This is a problem that does not fall in any of the standard categories. What is required is to understand the purpose and workings of the 'integrating factor' method. Memorized formulas will not apply. We want to multiply the ODE with whatever quantity is feasible such that the resulting ODE can be integrated, i.e., an antideriviative of either side can be found without prior knowledge of the function $y(x)$. Here, y'/y does the trick, because it turns the ODE into the form $y'^2 y'' = y'/y$, which is $\frac{d}{dx}(\frac{1}{3}y'^3) = \frac{d}{dx}\ln|y|$. So we can integrate and get the 1st order ODE $\frac{1}{3}y'^3 = \ln|y| + C$. This, as well as the further integration by separation of variables, which is now possible, was not asked. This is the only multiplier in the list that does the trick.

ANSWER: D

Pblm #19: For which choices of a real number a do all solutions to $y'' + 2ay' + (1-a)y = 0$ satisfy $\lim_{x\to\infty} y(x) = 0$?

Solution: The general solution is $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ where $r_{1,2}$ are the roots of the characteristic equation $r^2 + 2ar + (1-a) = 0$, provided these roots are distinct. (This is also true if $r_{1,2}$ are complex, even though we may wish to rewrite the complex exponentials into trigs in this case.)

By the quadratic formula, we have $r_{1,2} = -a \pm \sqrt{a^2 - (1-a)}$. Now if $a < 0$, then at least one of $r_{1,2}$ (namely the one that comes from choosing the + sign) is either positive or complex with positive real part. The corresponding solution $e^{r_1 x}$ does *not* go to 0, but goes to ∞. For $a = 0$, we get trig solutions that do not go to 0 either. If $0 < a < 1$, the square root in the quadratic formula is $< |a|$, so the negative sign in $-a$ surivives even if the square root is added (if the radicand is negative, r_1, r_2 have negative real part). We have exponentials that decay to 0 as $x \to \infty$. For $a = 1$, we have $r_2 = 0$, giving a constant solution $t = C_2$. Hence, not all solutions go to 0 as $x \to \infty$. (Some do, but the question asked when all do.) If $a > 1$ then the root with the + sign is positive and gives rise to a solution that goes to infinity.

ANSWER: D

Note: One can use Viete's formula instead: We are looking for those cases when either r_1 and r_2 are both negative, or else are complex with negative real part. In either case this requirement implies that $-2a = r_1 + r_2 < 0$, and $1 - a = r_1 r_2 > 0$. So we see immediatly that only those a that satisfy $0 < a < 1$ have a chance to satisfy the requirement. One now convinces oneself that these a indeed do satisfy the required condition.

Pblm #20: Solve the IVP

$$y' = \frac{4y^2 + 4xy + x^2}{4x^2} \quad y(1) = 0.$$

Solution: This is a homogeneous ODE, and it can therefore be handled by the substitution $v = y/x$. We get

$$v'x + v = \frac{4v^2 + 4v + 1}{4} \quad v(1) = 0$$

Simplifying and separation leads to

$$\int_0^y \frac{4dv}{4v^2+1} = \int_1^x \frac{dx}{x}$$
$$2\arctan(2v) = \ln x$$
$$y = xv = \frac{1}{2}x\tan(\frac{1}{2}\ln x)$$

ANSWER: A

Pblm #21: The parabolas $y = Cx^2$ are solutions of which ODE?

Solution: In principle we can try each answer and plug in $y = Cx^2$. Alternatively, we can take the derivative: $y' = 2Cx$ and see that $y' = \frac{2}{x}y$.

ANSWER: A

Pblm # 22: Which is the general solution to the ODE $y'' + 2y' - 3y = 0$?

Solution: Using the characteristic equation $r^2 + 2r - 3 = 0$ with the solutions $r = 1$ or $r = -3$, we get $y = C_1e^x + C_2e^{-3x}$.

ANSWER: E

Pblm #23: Use Euler's method with a single step for the ODE IVP $y' = (x+y)/(x^2+y^2)$, $y(1) = 2$, to find an approximation for $y(1.1)$. You get what?

Solution:

$y(1.1) \approx y(1) + 0.1y'(1)$ with $y'(1) = (1+2)/(1^2+2^2) = 3/5 = 0.6$. So $y(1.1) \approx 2.06$.

ANSWER: E

Pblm #24: For which choice of $g(x,y)$ is the ODE $(2x - 3y^2)\,dx + g(x,y)\,dy = 0$ exact?

Solution:

For the eqn to be exact, we need $\frac{\partial}{\partial y}(2x - 3y^2) = \frac{\partial}{\partial x}g(x,y)$, hence $g_x = -6y$ which is $g = -6xy + h(y)$.

ANSWER: D

Pblm #25: Solve the IVP $y' + xy = x$, $y(0) = 2$ and find $y(2)$.

Solution: An integrating factor μ must satisfy $x\mu = \mu'$, so we can take $\mu = e^{x^2/2}$. The ODE becomes $(e^{x^2/2}y)' = xe^{x^2/2}$. The general solution is $y = Ce^{-x^2/2} + 1$. The same result can also be obtained by adding the general solution to the homogeneous equation, which is separable, to a particular solution $y = 1$ (obtained by guessing) of the inhomogeneous equation. $y(0) = 2$ implies $C = 1$. Then $y(2) = 1 + e^{-2}$.

ANSWER: B

Pblm #26: Solve the IVP $xy' + y = 1$, $y(1) = 2$ and find $y(2)$.

Solution: This can be immediately integrated, by writing it as $(xy)' = 1$. The general solution is $xy = x + C$, or $y = 1 + \frac{C}{x}$. From $y(1) = 2$ we get $C = 1$. Then $y(2) = \frac{3}{2}$.

ANSWER: C

Pblm #27: If $y' = |1 - x|$ and $y(0) = 0$, what is $y(3)$?

Solution: By the fundamental theorem of calculus, $y(3) - y(0) = \int_0^3 |1 - x|\, dx = \int_0^1 (1 - x)\, dx + \int_1^3 (x - 1)\, dx = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$. (The integral can also be seen geometrically as the sum of two triangle areas) With $y(0) = 0$, we conclude

ANSWER: C

Pblm # 28 If y is a solution to

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}.$$

with $y(0) = -1$, find $y(1)$.

Solution:

$2(y - 1)dy = (3x^2 + 4x + 2)dx$. Use $y(0) = -1$ and do antideriviatives $y^2 - 2y = x^3 + 2x^2 + 2x + 3$. Use initial condition again and solve for y and get $y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$. $y(1) = -2$.

ANSWER: E. none of the above

#29: If x and y solve

$$x' = x + y$$

$$y' = 4x + y$$

and $x(0) = 1$ and $y(0) = 1$, find $x(1)$.

Solution: The eigenvalues of the matrix are $-1, 3$ or use $x'' = 2x' + 3x$. $x = ae^{3t} + be^{-t}$ and use $x(0) = 1, y(0) = 1$ to get $x'(0) = 2$. Then $a = 3/4$ and $b = 1/4$. $x(1) = 3e^3/4 + e^{-1}/4$

ANSWER: E. none of the above

Pblm #30: Johnny Appleseed Pike and John Hancock Highway are straight roads that intersect at a right angle. An SUV is heading, on Appleseed Pike, towards the intersection with speed 50mi/hr. A convertible is heading away from this intersection on Hancock Hwy at a speed of 60 mi/hr. At the moment when the SUV car is 1 mile before the intersection and the convertible is 1/2 mile past the intersection, at what rate is the distance between the cars changing? (Distance measured as the crow flies).

Solution: $x^2 + y^2 = d^2$ and then $2xx' + 2yy' = 2dd'$. Using $x = 1, x' = -50, y = 1/2, y' = 60, d = \sqrt{5}/2$, gives $d' = -8\sqrt{5}$

ANSWER A. decreasing at $8\sqrt{5}$ mi/hr