2009 Equations and Inequalities Theta Topic Test

Solutions

1. For positive roots, we have: +-++-, so three sign changes: 3 or 1 positive roots. For negative roots, we have -++--, so two sign changes: 2 or 0 negative roots. Choices are: 3+, 2-, 0i; 1+, 2-, 2i; 3+, 0-, 2i; 1+, 0-, 4i. The greatest number of possible imaginary roots is 4, **E**.

2. $4x+7<2x+50 \to 2x<43 \to x<21.5$. Only **D** does not belong in the solution set.

3. $x=\dfrac{8\pm\sqrt{64-4(2)(1)}}{2(2)}=\dfrac{8\pm 2\sqrt{14}}{4}=2\pm\dfrac{\sqrt{14}}{2}$. $\sqrt{14}$ is between 3 and 4: **A**.

4. Sum of the reciprocal of the roots is –(linear coefficient)/constant term: $\dfrac{-7}{-110}=\dfrac{7}{110}$, **A**.

5. Using expansion of minors, $3(2x^2)-x(-x^3)+6(-x^2-2)=7 \to x^4-19=0$. The product of the roots is 19. **E**.

6. $27^{5x+3}=81^{|x+2|-7} \to 3^{15x+9}=3^{4|x+2|-28} \to 4|x+2|=15x+37$. This gives $x+2=\dfrac{15}{4}x+\dfrac{37}{4}$ or $x+2=-\dfrac{15}{4}x-\dfrac{37}{4}$. The two solutions are $-\dfrac{29}{11}$ and $-\dfrac{45}{19}$, but $-\dfrac{29}{11}$ is extraneous: **D**.

7. $27^{|x+2|^2}=9^{|x+2|+2} \to 3^{3|x+2|^2}=3^{2|x+2|+4} \to 3|x+2|^2=2|x+2|+4 \to 3|x+2|^2-2|x+2|-4=0$.
$|x+2|=\dfrac{2\pm\sqrt{4-4(3)(-4)}}{2(3)}=\dfrac{1\pm\sqrt{13}}{3}$. Solving for x yields $\dfrac{-5\pm\sqrt{13}}{3}$ and $\dfrac{-7\pm\sqrt{13}}{3}$.
The only viable solutions are $\dfrac{-5+\sqrt{13}}{3}$ and $\dfrac{-7-\sqrt{13}}{3}$, whose sum is –4, **B**.

8. $xy=x-y \to x(y-1)=-y \to x=\dfrac{-y}{y-1}$. y cannot be 1, so correct answer is **E**.

9. The discriminant is 25, a perfect square, so the roots are rational, **C**.

10. $\dfrac{5}{3+x}=x \to 5=x^2+3x \to x=\dfrac{-3\pm\sqrt{9-4(1)(-5)}}{2(1)}=\dfrac{-3\pm\sqrt{29}}{3}$. Only the positive solution is feasible, **A**.

11. $\dfrac{1}{|3x+1|}\ge 5 \to |3x+1|\le\dfrac{1}{5} \to 3x+1\le\dfrac{1}{5}$ and $3x+1\ge-\dfrac{1}{5}$, $x\ne-\dfrac{1}{3}$. This results in $x\le-\dfrac{4}{15}\cap x\ge-\dfrac{2}{5}$: $\left[-\dfrac{2}{5},-\dfrac{1}{3}\right)\cup\left(-\dfrac{1}{3},-\dfrac{4}{15}\right]$, **E**.

12. $\dfrac{\frac{1}{4}}{\frac{1}{x}} = \dfrac{1}{x} \rightarrow \dfrac{1}{4} = \dfrac{1}{x^2} \rightarrow x = \pm 2$. We consider only the positive solution, and **C** is the answer.

13. Since both sides must be positive, squaring each side will retain the positive values: $4x^2 + 4x + 1 > x^2 - 10x + 25 \rightarrow 3x^2 + 14x - 24 > 0 \rightarrow (3x-4)(x+6) > 0$. This yields the intervals $(-\infty, -6) \cup \left(\frac{4}{3}, \infty\right)$. The closest integers that are also in these intervals are -7 and 2, whose sum is –5, **A**.

14. $4x^2 + 8x - 2\sqrt{4x^2 + 8x - 3} = 6 \rightarrow 4x^2 + 8x - 3 - 2\sqrt{4x^2 + 8x - 3} = 6 - 3$. Substituting $y = 4x^2 + 8x - 3$, we have $y - 2\sqrt{y} - 3 = 0 \rightarrow \left(\sqrt{y} - 3\right)\left(\sqrt{y} + 1\right) = 0 \rightarrow y = 9$. Now, $4x^2 + 8x - 3 = 9 \rightarrow (x+3)(x-1) = 0 \rightarrow x = -3,\ x = 1$. **E**.

15. $|x^2 - 5x| < 6 \rightarrow x^2 - 5x - 6 < 0$ and $x^2 - 5x + 6 > 0$. These two inequalities yield $(-1, 2)$ and $(-\infty, 2) \cup (3, \infty)$. The common solution is $(-1, 2) \cup (3, 6)$. The integers in the set are 0, 1, 4, and 5, whose sum is 10, **A**.

16. $\begin{cases} \log_x w = 36 \\ \log_y w = 18 \\ \log_{xyz} w^2 = 12 \end{cases} \rightarrow x^{36} = w,\ y^{18} = w.\quad \log_{xyz} w^2 = 12 \rightarrow 2\log_{xyz} w = 12 \rightarrow \log_{xyz} w = 6$.

$(xyz)^6 = w,\ x = w^{\frac{1}{36}},\ y = w^{\frac{1}{18}}$. Now, $\left(w^{\frac{1}{36}} w^{\frac{1}{18}} z\right)^6 = w \rightarrow w^{\frac{1}{2}} z^6 = w \rightarrow z^6 = w^{\frac{1}{2}} \rightarrow w = z^{12}$.

$\log_z w = 12$, **B**.

17. $\dfrac{\log(4x - 156)}{12x + 31} \le 0$ tells us that $x \ne -\dfrac{31}{12}$ and $x > 39$ (by definition of logs).

For the numerator to be positive and the denominator negative, we have $x > \dfrac{157}{4},\ x < \dfrac{31}{12}$.

For the numerator to be negative and the denominator positive, we have $x < \dfrac{157}{4}, x > -\dfrac{31}{12}$.

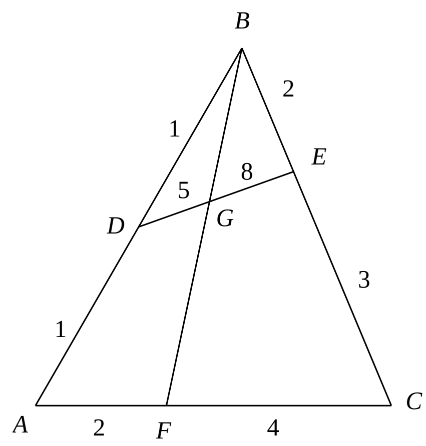
For the numerator to be 0, we have $x = \dfrac{157}{4}$. There is no solution for the positive numerator, negative denominator case. For the log expression to be defined, *x* must be greater than 39. The final solution set is $\left(39, \dfrac{157}{4}\right]$, the difference which is $\dfrac{1}{4}$, **D**.

18. The point *A*, the point of tangency, and the center of the circle determine a right triangle with legs of length *x* and *L* and hypotenuse of length $x + r$, where *x* is the shortest distance from

A to the circle. Using the Pythagorean theorem, $\left(\frac{4}{3}r\right)^2 + r^2 = (x+r)^2 \rightarrow x = \frac{2}{3}r = \frac{L}{2}$, **C**.

19. By assigning mass points, we can set up the following system of equations:

$$\begin{cases} 1A + 1B = 2D \quad \rightarrow \quad 4A + 4B = 8D \\ 2C + 3B = 5E \\ 5E + 8D = 13G \end{cases}$$



Combining the first two equations, we get $(4A+4B)+(2C+3B)=(8D)+(5E)$ which simplifies to $(4A+2C)+7B=8D+5E$.
Since we are using mass points, $4A+2C=6F$.
By substitution, $6F+7B=8D+5E$. Again, since we are using mass points, $8D+5E=13G$.
Using substitution again, we have $6F+7B=13G$,
So *BG*:*GF* is 6:7, **E**.

20. $3^x - 3^{x-3} = 78\sqrt{3} \rightarrow 3^x\left(1-3^{-3}\right) = 78\sqrt{3} \rightarrow \frac{26}{27} \cdot 3^x = 78\sqrt{3} \rightarrow 3^x = 81\sqrt{3} = 3^4 3^{\frac{1}{2}} = 3^{\frac{9}{2}}$, **E**.

21. $\frac{7}{10} = 0.7$ and $\frac{11}{15} \approx 0.733$, so no fraction with denominator of 1, 2, 3, 4, 5, or 6 will satisfy this condition. $\frac{5}{7}$ is the fraction with the smallest denominator that satisfies the inequality, so **D** is the correct choice.

22. *x* and *y* must both be positive in order for $x + y + xy = 54$. Trying $x = 2$ gives a non-integer value for *y*. Trying $x = 4$, we get $y = 10$. By symmetry, we also get (10, 4). No other positive integer pairs satisfy the equation, so **B**, 14, is the sum of *x* and *y*.

23. The graph of $y = |x-2| - 1$ has vertex (2, –1) which moves to (2, 1) on the graph of $y = \left||x-2|-1\right|$. Therefore, the line $y = 1$, **B**, will intersect the graph three times: on the two outermost oblique lines and at the vertex.

24. If the rabbit eats *x* kg of carrots per day, then $x + 365x = 111 \rightarrow x = \frac{111}{366} = \frac{37}{122}$, **C**.

25. Let *x* be the free weight allowance and let *y* (in dollars) be the excess weight charge, per kilogram, greater than *x*. 160 = (number of free kg)(cost of free kg) + (number of charged kg)(cost of charged kg) = (2*x*)(0) + (52 – 2*x*)*y*. Alone, Hope would have had 340 = (*x*)(0) + (52 – *x*)*y*. Combining these two equations, in terms of *y*, we have:
$y = \frac{160}{52-2x} = \frac{340}{52-x}$. Solving for *x*, we get $x = 18$, **A**.

26. $x^4 - 9x^2 + 4x + 12 < 0$ factors into $(x+3)(x+1)(x-2)^2 < 0$, whose solution set is $(-3, -1)$. $-3 + (-1) = -4$, **D**.

27. If x is the sufficient amount of bricks, then $0.93x = 10000 \to x = \frac{1000000}{93} \approx 10753$. 10,800 bricks, **A**, should be ordered.

28. Let the proportion of the circle the faster runner completes in 1 second be $\frac{1}{x}$ and the proportion for the slower runner be $\frac{1}{y}$. Then the times taken for a full lap are x and y seconds, respectively, so $y - x = 10$. In 12 minutes (720 seconds), the faster runner completes one extra lap, so we have $\frac{720}{x} - \frac{720}{y} = 1 \to \frac{720}{x} - \frac{720}{x+10} = 1$. Solving, the only feasible solution for x is $x = 80$, **A**.

29. Rewrite the given equations as $a + b = -c$ and $(a+b)\left(a^2 - ab + b^2\right) + c^3 = 216$. We also have $(a+b)^2 = (-c)^2 = a^2 + 2ab + b^2 = c^2$. By substitution and after simplifying we have $(-c)\left(c^2 - 3ab\right) + c^3 = 216$ which yields $3abc = 216 \to abc = 72$, **B**.

30. Begin with $y = mx$ substitution.

$$\begin{cases} x^2 + xy + 3y^2 = 15 \\ -5x^2 + 31xy - 3y^2 = 45 \end{cases} \to \begin{cases} x^2 + mx^2 + 3m^2x^2 = 15 \\ -5x^2 + 31mx^2 - 3m^2x^2 = 45 \end{cases} \to \begin{cases} 1 + m + 3m^2 = 15 \\ -5 + 31m - 3m^2 = 45 \end{cases}$$

After multiplying the top equation by 3 and equating the expressions, we have $-5 + 31m - 3m^2 = 3 + 3m + 9m^2 \to (3m-1)(m-2) = 0$, so $y = \frac{1}{3}x$ or $y = 2x$.

Substituting with $y = \frac{1}{3}x$, we get $x^2 + \frac{1}{3}x^2 + \frac{1}{3}x^2 = 15 \to x = \pm 3 \Rightarrow (3, 1), (-3, -1)$.
Substituting with $y = 2x$, we get $x^2 + 2x^2 + 12x^2 = 15 \to x = \pm 1 \Rightarrow (1, 2), (-1, -2)$.

The Quadrant I solutions are (3, 1) and (1, 2). The distance between them is $\sqrt{5}$, **C**.

TB1. $6241 = r^2 \to r = 79$. Therefore, the circumference is 158π.

TB2. The lines will intersect at $(0,2)$, $\left(\frac{4}{3}, -2\right)$, $\left(-\frac{4}{3}, -2\right)$. Two of the legs have length $\frac{4}{3}\sqrt{10}$ and the third leg has length $\frac{8}{3}$. The most descriptive name is isosceles triangle or acute isosceles triangle.

TB3. $1 = 10^6 x^{-\frac{3}{2}} = x^{\frac{1}{2}} = 10^2 \rightarrow x = 10000$

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