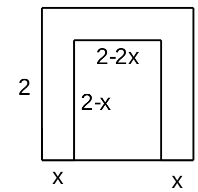
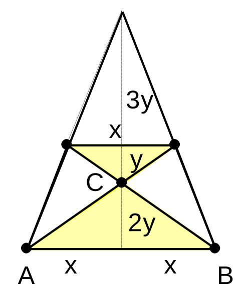
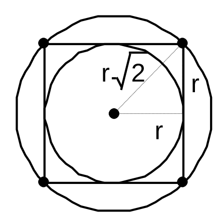
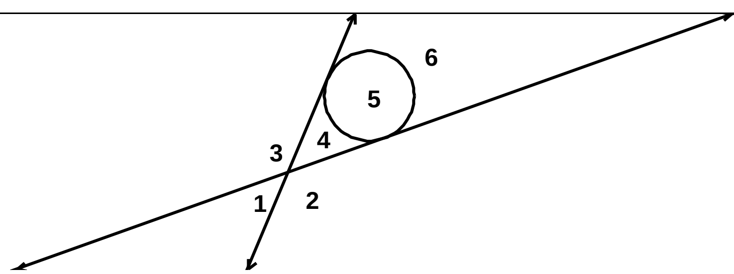
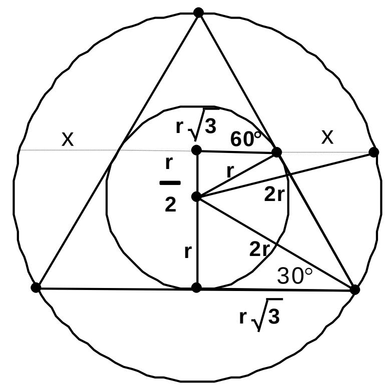
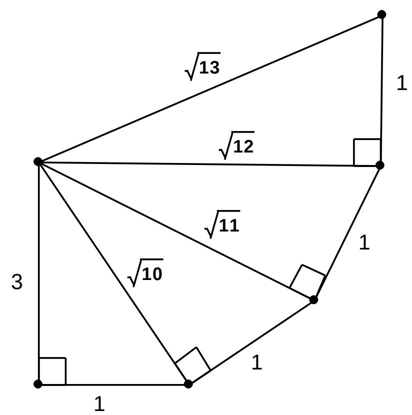
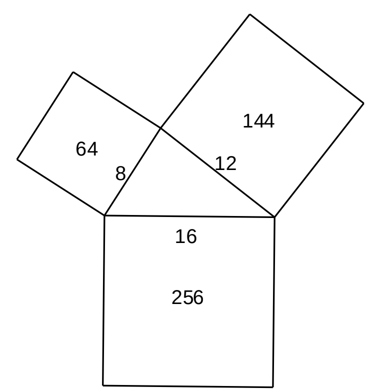
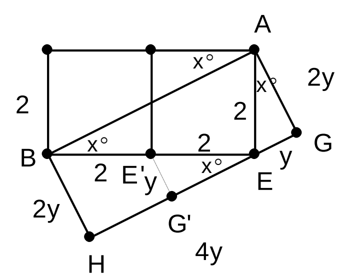
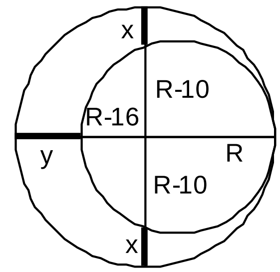
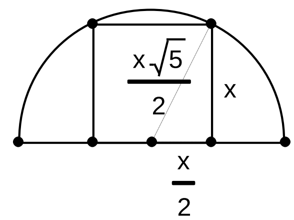
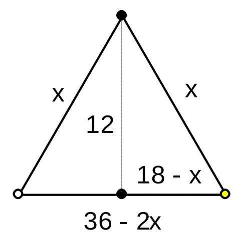
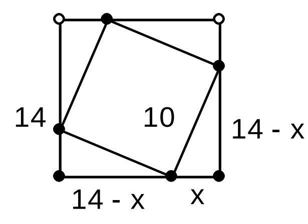
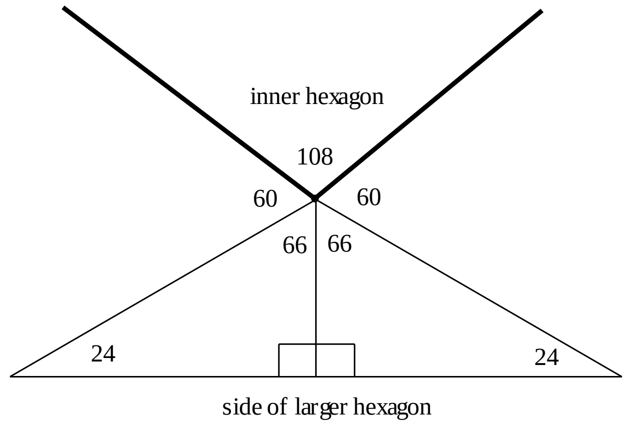
2009 Geometry Test (Theta)

Solutions:

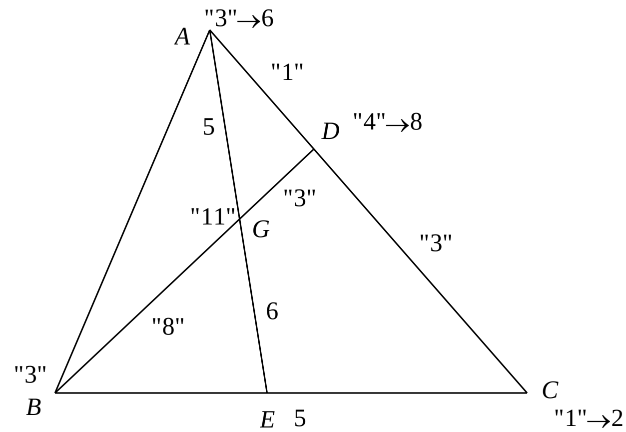
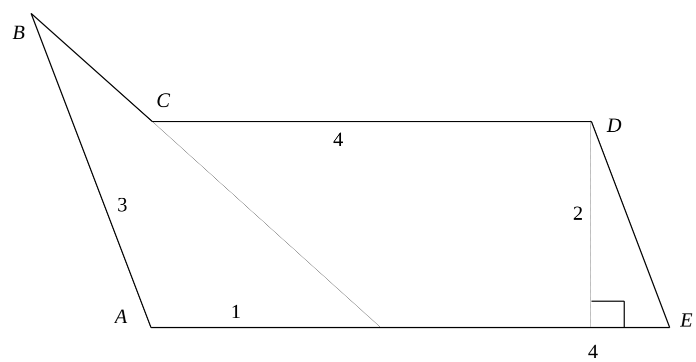
1.	**D**	In regular polygons of odd numbered sides the lines of symmetry are angle bisectors and their number is the same as the number of sides. In regular polygons of even numbered sides the lines of symmetry are angle bisectors and the perpendicular bisectors of sides and their number is the same as the number of sides.	
2.	**E**	Since this is a ratio problem it is appropriate to assign 2 as the length of a side of the square. That would mean that the square has area 4 and the rectangle has area 2. $(2 - 2x)(2 - x) = 2$ $4 - 6x + 2x^2 = 2$ $\qquad \frac{\text{band width}}{\text{square side}} = \frac{3 \pm \sqrt{5}}{4}$ $x^2 - 3x + 1 = 0$ $x = \frac{3 \pm \sqrt{5}}{2}$	
3.	**B**	The radius of a regular hexagon is equal to the length of its side. Therefore a hexagon inscribed in a circle has sides equal in length to the radius of the circle in which it is inscribed.	
4.	**B**	Area of original triangle – area of trapezoid = area cut off 60 – 45 = 15. Since the cut off triangle is similar to original the ratio of similitude is $\sqrt{15/60} = 1/2$. This makes the upper base of the trapezoid a mid-segment and the two shaded triangles also similar with a ratio of 1:2. Area of cut off triangle = .5(3xy) = 15. So xy = 10. Area of Δ ABC = x(2y) = 2xy = (by substitution) 2(10) = 20	
5.	**E**	$\frac{small}{annulus} = \frac{\pi r^2}{2\pi r^2 - \pi r^2} = \frac{\pi r^2}{\pi r^2} = 1$	
6.	**A**		

7.	**C**	Following the series of 30-60-90 triangles drawn in the adjacent sketch is becomes apparent that y is the mid-segment of the equilateral triangle and $y = r\sqrt{3}$. Using the Pythagorean theorem on the right triangle whose sides are $\frac{r}{2},\ 2r,\ \frac{y}{2}+x$ gives $\frac{y}{2}+x=\sqrt{4r^2-\frac{r^2}{4}}=\sqrt{\frac{15r^2}{4}}=\frac{r\sqrt{15}}{2}$ since we know y = $r\sqrt{3}$ $\frac{r\sqrt{3}}{2}+x=\frac{r\sqrt{15}}{2}$ $x=\frac{r\sqrt{15}}{2}-\frac{r\sqrt{3}}{2}=\frac{r\sqrt{15}-r\sqrt{3}}{2}$ It follows that $\frac{x}{y}=x\left(\frac{1}{y}\right)=\frac{r\sqrt{15}-r\sqrt{3}}{2}\left(\frac{1}{r\sqrt{3}}\right)=\frac{\sqrt{5}-1}{2}$	
8.	**C**		
9.	**D**	S = .5(8 + 12 + 16) = 18 $Area=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{18(10)(6)(2)}=12\sqrt{15}$	
10.	**A**	Note that △AGE is similar to△BEA and △EHB with a ratio of 1:2. Using the Pythagorean theorem on △AGE will give $y^2+4y^2=4\rightarrow 5y^2=4\rightarrow y^2=\frac{4}{5}$. The area of the right△BHE = $\frac{1}{2}(2y)(4y)=4y^2$ by substitution the area is $\frac{16}{5}$	

11.	**B**	Using the small circle: $R(R-16)=(R-10)(R-10)$ $R^2-16R=R^2-20R+100$ $4R=100$ $R=25$ The radius of the small circle is (2R-16)/2 which is R-8 = 17. Area = $17^2\pi=289\pi$	
12.	**C**	$\frac{x}{2r}=\frac{x}{x\sqrt{5}}=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$	
13.	**C**	Let the radius of the can be r; then it follows the height is 6r. The volume of the can is Bh = $\pi r^2 h=\pi r^2(6r)=6\pi r^3$. The volume of the three balls = $3\left(\frac{4}{3}\pi r^3\right)=4\pi r^3$. The unoccupied area is $6\pi r^3-4\pi r^3=2\pi r^3$. The fraction we want is $\frac{2\pi r^3}{6\pi r^3}=\frac{1}{3}$	
14.	**A**	$12^2+(18-x)^2=x^2$ $144+324-36x+x^2=x^2$ $468=36x$ $13=x$ $Area=(18-x)(12)=(5)(12)=60$	
15.	**E**	Let x = side of cube. Space diagonal = volume. $x\sqrt{3}=x^3$; $\sqrt{3}=x^2$; $\sqrt[4]{3}=x$	
16.	**E**	$x^2+(14-x)^2=100$; $x^2+196-28x+x^2=100$; $2x^2-28x+196=100$; $2x^2-28x+96=0$; $x^2-14x+48=0$ $(x-6)(x-8)=0$; $x=6$ $x=8$ **But it doesn't say which outer or inner vertex and could be a diagonal length.**	
17.	**D**	This is an example of an arithmetic series, the sum of which is the entire length of paper: 4(300) = 1200 inches. The first term a_1 is the circumference around the core: 2π. The last term a_n is the final circumference around the tissue: 6π; *n* is the number of times the tissue is wrapped around the roll. $S=\frac{n}{2}(a_1+a_n)$; $1200=\frac{n}{2}(2\pi+6\pi)$; $2400=n(8\pi)$; $\frac{300}{\pi}=n$ $\frac{300}{22/7}=\frac{7(300)}{22}=\frac{7(150)}{11}\approx 96$	

18.	**C**	The Euler line *always* contains the orthocenter, circumcenter, centroid, and center of the nine-point circle. The incenter is on the Euler line if the triangle is isosceles.
19.	**B**	Doubling the square is not one of the famous impossible constructions.
20.	**E**	Each interior angle of a regular hexagon measures 108. Each interior angle of an equilateral triangle measures 60. Subtracting from 360 and dividing by 2, we see that the base angles of the isosceles triangle is 24, and this is the angle we must use if cosine is involved. 
21.	**B**	If *x* be the fractional part of the circle that contains the sector in question, then $49 = x\pi r^2$. This fractional part is also the same fractional part of the circumference that is contained by this sector, so $2\pi rf =$ arc length, *s*. Dividing these equations, we get $\frac{s}{49} = \frac{2}{r}$, so $sr = 98$. The perimeter of the sector is 28 cm. $28 = s + 2r = s + 2\left(\frac{98}{s}\right) \to s^2 - 28s + 196 = 0 = (s-14)^2$. The arc length *s* must be 14.
22.	**D**	If the triangle in question is $\triangle ABC$ and the point in question is *P*, then we want the area of $\triangle PAB = \frac{1}{3}$area of $\triangle ABC$. If *d* is the distance from *P* to side *c*, and *h* is the distance from *C* to *c*, then $\frac{1}{2}dc = \frac{1}{3}\left(\frac{1}{2}ch\right) \Rightarrow d = \frac{1}{3}h$, a property of medians.
23.	**E**	The diagonal mentioned in the problem divides the rhombus into two equilateral triangles. Using the formula for the area of an equilateral triangle, given the side length, we can find the area of the rhombus: $2\left(\frac{\left(\frac{3}{16}\right)^2\sqrt{3}}{4}\right) = \frac{9\sqrt{3}}{512}$ in^2. Now, 400mi/1.5in can be coverted to 640000/9 mi^2/in^2. $\frac{9\sqrt{3}}{512} \square \frac{640000}{9} = 1250\sqrt{3}$.
24.	**B**	Let the upper segments of each side be *x* and *y*. Setting the upper and lower perimeters equal, we have $x + 3 + y = (4 - x) + 9 + (6 - y) \to 2x + 2y = 16 \to x + y = 8$. We know that $\frac{x}{y} = \frac{4}{6} = \frac{2}{3}$, so $x = \frac{2y}{3}$. By substitution we have $\frac{2y}{3} + y = 8 \to y = \frac{24}{5}$. Now we know that $6 - y = \frac{6}{5}$. The ratio of parts on the "*x*-side" is the same as the ratio on the "*y*-side," so

		$\frac{24}{5}:\frac{6}{5}\to\frac{4}{1}$.
25.	**D**	The problem can probably be solved more easily by placing it on a coordinate grid with *C* at the origin. This places *A* at (0, 9), *Y* at (0, 4), *X* at (4, 0), and *B* at (9, 0). The equation for $\overleftrightarrow{BY}$ is $y=-\frac{4}{9}x+4$ and the equation for $\overleftrightarrow{AX}$ is $y=-\frac{9}{4}x+9$. These two lines intersect at $\left(\frac{36}{13},\frac{36}{13}\right)$. The area of $\triangle ABD = \triangle ABC-(\triangle ACX+\triangle BDX)=$ $\frac{1}{2}(9)(9)-\left(\frac{1}{2}(9)(4)+\frac{1}{2}\left(\frac{36}{13}\right)(5)\right)=\frac{81}{2}-\frac{324}{13}=\frac{405}{26}$.
26.	**D**	The maximum number of pieces resulting from *n* planar cuts through a cylinder is called a cake number, and the number can be found by $\frac{1}{6}\left(n^3+5n+6\right)\to\frac{125+25+6}{6}=\frac{156}{6}=26$.
27.	**C**	We can begin by labeling a basic equilateral triangle, breaking it into two 30-60-90 triangles. The height of the triangle will be the altitude of the face of the tetrahedron, $\frac{s\sqrt{3}}{2}$. This will also be the "hypotenuse" for the sine value we are looking for. The center of the triangle is the centroid. Since we are dealing with the dihedral angle, we need the distance to the centroid from the side of the triangle. This distance is one-third of the length of the altitude, $\frac{s\sqrt{3}}{6}$. The Pythagorean theorem will give us the altitude of the tetrahedron, which we will need to find the sine value (it will be the "opposite"): $a^2=\frac{3}{4}s^2-\frac{3}{36}s^2=\frac{24}{36}s^2\to a=\frac{s\sqrt{6}}{3}$. Sine of the dihedral angle = $\frac{\frac{s\sqrt{6}}{3}}{\frac{s\sqrt{3}}{2}}=\frac{2\sqrt{6}}{3\sqrt{3}}=\frac{2}{3}\sqrt{2}$.
28.	**A**	The smallest angle is across from the "4" side. Use the law of cosines: $\cos\theta=\frac{8^2+6^2-4^2}{2(8)(6)}=\frac{7}{8}$.
29.	**B**	If the chords intercept a 120-degree arc, then they meet to form a 60-degree inscribed angle. Using the law of cosines and *x* to represent the length of the chord that intercepts the 120-degree arc, we have $x^2=10^2+8^2-2(10)(8)\cos 60=84\to x=2\sqrt{21}$. Using the extended law of sines, we have $2R=\frac{2\sqrt{21}}{\sin 60}\to R=2\sqrt{7}$, where *R* represents the radius of the circumscribed circle.

30.	**D**	Using mass points, we quickly see that we have "overlap" at points *A*, *C*, and *D*. Doubling those numbers will give us the correct ratios. 
TB1.	**1**	The triangular numbers begin with 1, 3, 6, 10, 15 . . .
TB2.	**9.5**	Extend $\overline{BC}$ to form the hypotenuse of a right triangle. (The diagram was not to scale!) The area of the triangle is 1.5. The area of the resulting parallelogram is 8. The total area of the pentagon is 9.5. 
TB3.	$\frac{\mathbf{7}}{\mathbf{3}}\pi$	One way to work the problem is to use the law of cosines twice, once in each triangle shown at the right. If the sides of length 1 are named *a* and the sides of length 2 are named *b*, then $3a+3b=360 \rightarrow a+b=120$. The central angle has measure 120 degrees and the inscribed angle will also have measure 120 degrees. Now, with the law of cosines, we have $x^2=1^2+2^2-2(1)(2)\cos 120=7$ and $x^2=r^2+r^2-2r^2\cos 120=3r^2$. $3r^2=7$, so $\pi r^2=\frac{7}{3}\pi$.