For all questions, answer choice "E) NOTA" means none of the above answers is correct.

1. Evaluate: $\lim_{k\to 4}\frac{k^2-2k-8}{k-4}$

A) 2 B) 6 C) 8 D) does not exist E) NOTA

2. Evaluate: $\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{x^2}}{e^x}$

A) $1/\sqrt{e}$ B) 1 C) $\sqrt{e}$ D) ∞ E) NOTA

3. For $y=x^{x+x^{x+x^{x+\cdots}}}$, evaluate $\frac{dy}{dx}$ at the point $(1,1)$.

A) $1/2$ B) 1 C) 2 D) 4 E) NOTA

4. For $y=x+x^{x+x^{x+x^{x+\cdots}}}$, find the value of $\lim_{x\to 0^+}x^y$.

A) 0 B) $1/\sqrt{e}$ C) 1 D) ∞ E) NOTA

5. The radius of a circular ink spot is $\frac{1+4t}{1+t}$ cm, t seconds after it appears. Find the rate of increase of the area of the ink spot (in cm^2/sec) when the radius is twice its initial value.

A) $4\pi/3$ B) $8\pi/3$ C) $16\pi/3$ D) 8π E) NOTA

6. Two lines tangent to the graph of $y=4x-x^2$ pass through the point $(2,5)$. Find the product of their slopes.

A) -16 B) -12 C) -9 D) -4 E) NOTA

7. Let a and b be real numbers such that $a>b>1$. Compute the value of $\lim_{n\to\infty}\left(a^n-b^n\right)^{1/n}$.

A) a B) b C) $\ln a$ D) $\ln b$ E) NOTA

8. Evaluate: $\lim_{x\to 0}\frac{2\sin x-\sin 2x}{3\sin x-\sin 3x}$

A) $\frac{1}{4}$ B) $\frac{8}{27}$ C) $\frac{4}{9}$ D) $\frac{2}{3}$ E) NOTA

9. Let $M=\begin{bmatrix} x & 2x & 3x \\ f(x) & g(x) & h(x) \\ 0 & 1 & 1 \end{bmatrix}$. If $f(x)=\ln\left(e^x+1\right)$ and $g(x)=\ln\left(e^x-1\right)$, find the value of $h'(\ln 3)$, assuming the value of the determinant of M is 0.

A) $\frac{9}{8}$ B) $\frac{9}{4}$ C) 3 D) 6 E) NOTA

10. Here is an actual example used in the text in which l'Hôpital's Rule was first published:

For positive real a, compute the value of $\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$.

A) $-\frac{4a}{3}$ B) $-a$ C) $\frac{3a}{4}$ D) $\frac{16a}{9}$ E) NOTA

For questions 11-12, use the following scenario: An aerosol can is designed as follows: the base is a cylinder of radius length r and height h, and the "lid" of the can is a hemisphere with radius length r on top of the base (imagine what a silo looks like). Let V and S be the volume and surface area of the can, respectively.

11. When $h=1.5$, the slant asymptote of $y=\frac{V}{S}$ is the line $y=ar+b$. Evaluate $\frac{b}{a}$.

A) $\frac{1}{2}$ B) $\frac{5}{6}$ C) $\frac{4}{3}$ D) $\frac{5}{2}$ E) NOTA

12. Given the can has surface area $80\pi\ \text{m}^2$, what is the maximal volume of this can, in m^3?

A) $\frac{160\pi}{3}$ B) 80π C) $\frac{320\pi}{3}$ D) 160π E) NOTA

13. The graph of $f(x)=x^3-3x^2-45x+120$ has two critical points at the points (a_1,b_1) and (a_2,b_2). Evaluate $a_1a_2+b_1+b_2$.

A) -72 B) 22 C) 131 D) 264 E) NOTA

14. Let $f(x)=\sum_{k=1}^{\infty}\left(1-\sqrt{x}\right)^k$ for values of x where the series converges. Where defined, find the minimum value of $\frac{f(x)}{f'(x)}$.

A) $-\frac{2}{9}$ B) $\frac{4}{9}$ C) 4 D) $-\frac{8}{27}$ E) NOTA

15. Consider the pentagon with vertices at the points $(-1,0)$, $(-1,1)$, $(0,2)$, $(1,1)$, and $(1,0)$, connected in that order. If the vertex initially at the point $(0,2)$ is moving up and to the right, parallel to the line $y=x$, at 2 units per second, at what rate, in square units per second, if the area of the pentagon changing after four seconds?

A) $\frac{\sqrt{2}}{2}$ B) $\sqrt{2}$ C) 2 D) $2\sqrt{2}$ E) NOTA

16. Using a tangent line approximation for the function $y=\sin x$ at the point where $x=\frac{\pi}{6}$, estimate the value of $\sin\left(\frac{\pi}{8}\right)$.

A) $\frac{24-\pi\sqrt{3}}{48}$ B) $\frac{24-\pi}{48}$ C) $\frac{24-\pi\sqrt{3}}{24}$ D) $\frac{24-\pi}{24}$ E) NOTA

17. Let $f(x)=\cfrac{a}{x+\cfrac{a}{x+\cfrac{a}{x+\ldots}}}$ and $g(x)=\cfrac{b}{x+\cfrac{b}{x+\cfrac{b}{x+\ldots}}}$, where a and b are positive constants with $a\neq b$. Evaluate $\lim_{x\to\infty}\frac{f(x)+\frac{x}{2}}{g(x)+\frac{x}{2}}$.

A) 0 B) 1 C) $\frac{a}{b}$ D) $\frac{b}{a}$ E) NOTA

18. Define $f(x)$ and $g(x)$ as in the previous problem. Assuming $a>b$, what is the minimum possible value of $f(x)-g(x)$?

A) 0 B) $\sqrt{a}-\sqrt{b}$ C) $2\left(\sqrt{a}-\sqrt{b}\right)$ D) $\sqrt{a}+\sqrt{b}$ E) NOTA

19. Suppose $x^3y + xy^2 + x + y = 9$. Find the sum of all possible values of $\frac{dy}{dx}$ at points where $x = 1$.

A) -2 B) -1 C) 0 D) 1 E) NOTA

Consider the following for questions 20-22: Hopefully you know how to use l'Hôpital's Rule! You might not, however, be familiar with the proof. Question 20 deals with the case where both the numerator, $f(x)$, and the denominator, $g(x)$, are assumed to be differentiable at the limit point; questions 21-22 deal with cases where this may or may not be true but is not assumed. In all three questions, assume that $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists.

20. Assume that $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$. Since f and g are differentiable at $x = c$, we know that $f(c) = g(c) = 0$. Let's therefore write $\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)}$. We can then divide by a certain quantity and recognize it as a famous definition or application of a theorem, after which we'll take a limit which completes the proof. If no other steps are involved, then what is the definition or theorem to which reference is being made?

A) Squeeze Theorem B) Difference Quotient C) Rolle's Theorem D) Mean Value Theorem E) NOTA

21. Let's once again assume that $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$, but let's no longer assume that $f(x)$ and $g(x)$ are differentiable at $x = c$. Now, redefine $f(c) = g(c) = 0$. This makes f and g continuous at $x = c$ but does not affect the limit since the function values do not matter as far as the limit is concerned. Since $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, there must be some interval $(c - \delta, c + \delta)$ such that for all x in the interval, except possibly $x = c$, both $f'(x)$ and $g'(x)$ exist, and $g'(x) \neq 0$. For $c < x < c + \delta$, the mean value theorem implies that $g(x) \neq 0$ and Cauchy's mean value theorem implies that there is a number ξ in the interval (c, x) such that $\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$. The next step is to take a limit as x approaches something. What theorem or definition do we then use to finish the proof?

A) Squeeze Theorem B) Difference Quotient C) Rolle's Theorem D) Mean Value Theorem E) NOTA

22. The case where $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ requires a lot of algebra—probably more than what's fair for this test (you are urged to go look it up afterward, though). That said, why can we not just use the proof in question 21?

A) $f(x)$ and $g(x)$ might not be continuous
B) replacing $f(c)$ and $g(c)$ with $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ no longer has any use
C) cannot use mean value theorem D) division by 0 happens
E) NOTA

23. Evaluate: $\lim_{x\to 0}\dfrac{2\sin x - \sin 2x}{x^2}$

A) -2 B) -1 C) 0 D) 2 E) NOTA

24. A spherical balloon is being inflated at a rate of 3 cm^3/sec. Find the rate of change of its surface area, in cm^2/sec, when its radius is 10 cm.

A) $3/5$ B) $5/4$ C) $3/2$ D) $\dfrac{6\sqrt{\pi}}{5}$ E) NOTA

25. Evaluate: $\lim_{x\to 0}\dfrac{\sin x - \tan x}{\sin^2 x}$

A) -1.5 B) -1 C) 0 D) 1 E) NOTA

26. Given that $f(x)=\dfrac{4-x}{2x+3}$ for $x>0$, find the maximum value of $f(x)+f'(x)$.

A) $3/16$ B) $3/8$ C) $1/2$ D) $11/16$ E) NOTA

Consider the following for questions 27-30: The supremum of a set is the smallest number greater than or equal to all the numbers in the set. The infimum of a set is the largest number less than or equal to all the numbers in the set. The notations for these concepts with reference to set A are $\sup(A)$ and $\inf(A)$, respectively. The infimum and supremum of a set do not necessarily have to be in the set itself (this applies only for infinite sets; however, in finite sets, the infimum is the minimum of the set and the supremum is the maximum of the set). Further, the limit inferior of a sequence a_n, denoted $\liminf a_n$, is defined as $\lim_{n\to\infty}\left(\inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}\right)$, and the limit superior of a sequence a_n, denoted $\limsup a_n$, is defined as $\lim_{n\to\infty}\left(\sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}\right)$.

27. If $a_n = \dfrac{1}{n} + (-1)^n$, what is the value of $\left(\liminf a_n\right) + \left(\limsup a_n\right)$?

A) -1 B) 0 C) 1 D) 2 E) NOTA

28. Consider the function $f(x) = 2x^2 - x$ on the domain $(-1,1)$. Find the supremum of the range of f.

A) 0.125 B) 1 C) 3 D) does not exist E) NOTA

29. Which of the following statements are true?

I) For sequences a_n and b_n, if $\limsup a_n = a$ and $\limsup b_n = b$, where a and b are finite, then $\limsup(a_n + b_n) = a + b$.

II) $\limsup a_n = \liminf a_n$ if and only if the sequence a_n converges.

III) $\liminf a_n \geq \inf_{n\geq 1}\{a_n\}$

A) I & II only B) I & III only C) II & III only D) I, II & III E) NOTA

30. Let $a_n = |\sin n|^{1/n}$. Find the value of $\liminf a_n$.

A) 0 B) $\pi/12$ C) $1/2$ D) 1 E) NOTA