ANSWERS

(1) C A A A B	**(6)** B D D A B	**(11)** C D D A A
(16) D B A A C	**(21)** D B C B C	**(26)** D C B C C

SOLUTIONS

1. We have

$$\begin{aligned}\sqrt{-1}\cdot\sqrt{-3}\cdot\sqrt{-11}\cdot\sqrt{-61} &= i\cdot i\cdot i\cdot i\cdot\sqrt{1\cdot 3\cdot 11\cdot 61}\\ &= i^4\sqrt{2013}\\ &= \sqrt{2013},\ \boxed{\text{C.}}\end{aligned}$$

2. The powers of i cycle between $i, -1, -i,$ and $1.$ Thus, we have

$$\begin{aligned}i^{2013} &= \left(i^4\right)^{503}\cdot i\\ &= i,\ \boxed{\text{A.}}\end{aligned}$$

3. The absolute value of the entire fraction is the ratio of the absolute values of the numerator and denominator. Using this, we have

$$\begin{aligned}\left|\frac{3+4i}{5+12i}\right| &= \frac{|3+4i|}{|5+12i|}\\ &= \frac{5}{13},\ \boxed{\text{A.}}\end{aligned}$$

4. We let x equal the expression we wish to evaluate. With a substitution, we obtain $x=\sqrt{\frac{i}{4}+\sqrt{\frac{i}{4}+\sqrt{\frac{i}{4}+\cdots}}}=\sqrt{\frac{i}{4}+x}$. Solving this equation with the quadratic formula gives

$$x^2-x-\frac{i}{4}=0 \Rightarrow x=\frac{1\pm\sqrt{1+i}}{2}.$$

Now, we must evaluate $\sqrt{1+i}$. We can write this in cis form as $1+i=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$. To take the square root of this, we utilize de Moivre's Theorem to obtain

$$\begin{aligned}\left[\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right]^{1/2} &= 2^{1/4}\operatorname{cis}\left(\frac{1}{2}\left(\frac{\pi}{4}+2\pi k\right)\right), k=0,1\\ &= 2^{1/4}\operatorname{cis}\left(\frac{\pi}{8}+\pi k\right), k=0,1.\end{aligned}$$

Combining this with the rest of the solution gives

$$\begin{aligned}x &= \frac{1\pm 2^{1/4}\operatorname{cis}\left(\frac{\pi}{8}+\pi k\right)}{2}, k=0,1\\ &= \frac{1}{2}+2^{-3/4}\operatorname{cis}\left(\frac{\pi}{8}+\pi k\right), k=0,1, \boxed{\text{A.}}\end{aligned}$$

5. Note that $\operatorname{Re}(\operatorname{cis}\theta)=\cos\theta$ and $\operatorname{Im}(\operatorname{cis}\theta)=\sin\theta$. Thus, we have

$$\begin{aligned}\frac{\prod_{n=1}^{45}\operatorname{Re}\left[\operatorname{cis}((2n-1)^\circ)\right]}{\prod_{n=1}^{45}\operatorname{Im}\left[\operatorname{cis}(2(2n-1)^\circ)\right]} &= \frac{\cos 1^\circ\cos 3^\circ\cdots\cos 89^\circ}{\sin 2^\circ\sin 6^\circ\cdots\sin 178^\circ}\\ &= \frac{\cos 1^\circ\cos 3^\circ\cdots\cos 89^\circ}{(2\sin 1^\circ\cos 1^\circ)(2\sin 3^\circ\cos 3^\circ)\cdots(2\sin 89^\circ\cos 89^\circ)}\\ &= \frac{1}{2^{45}}\left(\frac{1}{\sin 1^\circ\sin 3^\circ\cdots\sin 89^\circ}\right).\end{aligned}$$

The bottom expression can be written as

$$\begin{aligned}\sin 1^\circ \sin 3^\circ \cdots \sin 89^\circ &= \frac{\sin 1^\circ \sin 2^\circ \sin 3^\circ \cdots \sin 89^\circ}{\sin 2^\circ \sin 4^\circ \cdots \sin 88^\circ} \\ &= \frac{\sin 1^\circ \sin 2^\circ \sin 3^\circ \cdots \sin 89^\circ}{(2\sin 1^\circ \cos 1^\circ)(2\sin 2^\circ \cos 2^\circ)\cdots(2\sin 44^\circ \cos 44^\circ)} \\ &= \frac{1}{2^{44}}\left(\sin 45^\circ \frac{\sin 46^\circ \sin 47^\circ \cdots \sin 89^\circ}{\cos 1^\circ \cos 2^\circ \cdots \cos 44^\circ}\right) \\ &= \frac{\sqrt{2}}{2^{45}}\left(\frac{\sin 46^\circ \sin 47^\circ \cdots \sin 89^\circ}{\sin 89^\circ \sin 88^\circ \cdots \sin 46^\circ}\right) \\ &= 2^{-89/2},\end{aligned}$$

where we have used the fact that $\sin(90^\circ - \theta) = \cos\theta$. Our answer is then

$$2^{-45}\left(\frac{1}{2^{-89/2}}\right) = 2^{-1/2}, \ \boxed{\text{B.}}$$

6. The powers of i contain two sets of numbers that are additive inverses of each other, namely $(1,-1)$ and $(i,-i)$. Thus the only sets of four numbers that will satisfy $a=0$ are permutations of either $(1,1,-1,-1)$, $(i,i,-i,-i)$, and $(i,-i,1,-1)$. The first two have $\binom{4}{2}=6$ distinct arrangements each, while the last has $4!=24$ total arrangements, giving $2(6)+24=36$ overall. There are $4^4=256$ possibilities, giving a probability of $\frac{36}{256}=\frac{9}{64}$, $\boxed{\text{B.}}$

7. The solutions to the equation z_k form a hexagon in the complex plane, similar to the 6th roots of unity, except the side length of the hexagon is $\sqrt[6]{729}=3$. Thus $|z_3 - z_6|$ is equal to the distance between two diagonally opposite points on the hexagon. This is simply $2(3)=6$, $\boxed{\text{D.}}$

8. We have $v_1 = \langle a,b\rangle$ and $v_2 = \langle c,d\rangle$, giving $v_1 \cdot v_2 = ac+bd$. Intuition would lead us to try $\text{Re}(z\cdot w) = ac-bd$. This, however, is the conjugate of what we wish to obtain. Naturally, we would then take the conjugate of either z or w. This gives us $\text{Re}(\overline{z}\cdot w) = \text{Re}((ac+bd)+i(ad-bc)) = ac+bd$, $\boxed{\text{D.}}$

9. Going by the definition, we have

$$\begin{aligned}\binom{i}{4} &= \frac{i(i-1)(i-2)(i-3)}{4!}\\ &= -\frac{10}{24}\\ &= -\frac{5}{12},\ \boxed{\text{A.}}\end{aligned}$$

10. We have

$$\begin{aligned}2(\text{cis}43^\circ \otimes \text{cis}35^\circ) &= 2\cos 35^\circ \text{cis}43^\circ\\ &= 2\cos 35^\circ\left(\cos 43^\circ + i\sin 43^\circ\right)\\ &= 2\cos 35^\circ \cos 43^\circ + i\left(2\cos 35^\circ \sin 43^\circ\right)\\ &= \left(\cos(43^\circ+35^\circ)+\cos(43^\circ-35^\circ)\right) + i\left(\sin(43^\circ+35^\circ)+\sin(43^\circ-35^\circ)\right)\\ &= \left(\cos 78^\circ + i\sin 78^\circ\right) + \left(\cos 8^\circ + i\sin 8^\circ\right)\\ &= \text{cis}78^\circ + \text{cis}8^\circ.\end{aligned}$$

Thus, we have $\theta\varphi = (78)(8) = 624,$ $\boxed{\text{B.}}$

11. Note that we can rewrite the equation as $(a-6)^2+(b-3)^2=64,$ or the equation for a circle. If we were to convert z to the Cartesian plane, we would simply write $z=(x,y)=(a,b).$ Hence, R is a circle with radius $8,$ and thus has an area of $8^2 = 64\pi = \boxed{\text{C.}}$

12. Let $z = a+bi.$ Then we have

$$\begin{aligned}\left|z-|z|\right| &= \left|a+bi-|a+bi|\right|\\ &= \left|a+bi-\sqrt{a^2+b^2}\right|\\ &= \left|\left(a-\sqrt{a^2+b^2}\right)+bi\right|\\ &= \sqrt{a^2-2a\sqrt{a^2+b^2}+\left(a^2+b^2\right)+b^2}\\ &= \sqrt{2\left(a^2+b^2\right)-2a\sqrt{a^2+b^2}}.\end{aligned}$$

Now, since $|z| = \sqrt{a^2+b^2},$ this becomes

$$\sqrt{2|z|^2 - 2a|z|} = \sqrt{2}$$
$$\Rightarrow |z|^2 - a|z| - 1 = 0.$$

Using the quadratic formula, we solve for $|z|$ as $|z| = \dfrac{a \pm \sqrt{a^2+4}}{2}$. Since $\sqrt{a^2+4} > 2$, we take $|z| = \dfrac{a + \sqrt{a^2+4}}{2}$, $\boxed{\text{D.}}$

13. We have

$$\begin{aligned} v_1 \cdot v_2 &= x(1+i) + y(3+2i) \\ &= (x+3y) + i(x+2y) \\ &= 5+6i. \end{aligned}$$

This gives us the systems of equations

$$\begin{aligned} x+3y &= 5 \\ x+2y &= 6 \end{aligned}$$

which we solve as $(x, y) = (8, -1)$, which gives $x + y = 7$, $\boxed{\text{D.}}$

14. We have $B = A - \lambda I = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$. Thus,

$$\begin{aligned} \det(B) &= (3-\lambda)(-1-\lambda) - (-2)(4) \\ &= -3 - 3\lambda + \lambda + \lambda^2 + 8 \\ &= \lambda^2 - 2\lambda + 5 \\ &= 0. \end{aligned}$$

Solving gives $\lambda = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$, $\boxed{\text{A.}}$

15. It is clear that each $f_n(x)$ will be a polynomial of degree 4, since the roots are the vertices of a square. Now, note that each set of roots is a rotation of $\frac{\pi}{4}$ radians counterclockwise from the previous set of roots and furthermore, each set of roots has $\frac{1}{\sqrt{2}}$ times the amplitude of the previous set of roots. We began with the fourth roots of unity, which are $\text{cis}\left(\frac{\pi k}{2}\right)$, $0 \le k \le 3$. This means the nth set of roots are $\left(\frac{1}{\sqrt{2}}\right)^{n-1} \text{cis}\left(\frac{\pi}{4}(n-1)+\frac{\pi k}{2}\right) = \left[\left(\frac{1}{4}\right)^{n-1} \text{cis}\left(\pi(n-1)\right)\right]^{1/4}$. Of course, we can write this as $x^4 = \left[\left(\frac{1}{4}\right)^{n-1} \text{cis}\left(\pi(n-1)\right)\right] = (-1)^{n-1}\left(\frac{1}{4}\right)^{n-1} = \left(-\frac{1}{4}\right)^{n-1}$. This implies that $f_n(x) = x^4 - \left(-\frac{1}{4}\right)^{n-1}$.

Thus we have

$$\sum_{n=1}^{\infty} f_n(0) = -\sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^{n-1} = -\frac{1}{1+\frac{1}{4}} = -\frac{4}{5}, \ \boxed{\text{A.}}$$

16. The function will not intersect the x-axis when it has imaginary roots. This requires that the discriminant be less than 0. We have

$$5^2 - (4)(k^2)(9) < 0 \Rightarrow k^2 > \frac{25}{36} \Rightarrow k \in \left(-\infty, -\frac{5}{6}\right) \cup \left(\frac{5}{6}, \infty\right), \ \boxed{\text{D.}}$$

17. Let the roots be $r_1, r_2, \ldots, r_{2013}$, where $r_1 = 1$. The sum of the roots taken two at a time can be written as $\sum_{cyc} r_i r_j$, $0 < i, j \le 2013, i \ne j$. This can be written as

$$\sum_{cyc} r_i r_j = \sum_{cyc} r_1 r_a + \sum_{cyc} r_b r_c = \sum_{cyc} r_a + \sum_{cyc} r_b r_c,$$

Since $r_1 = 1$. We can see that this summation contains both the sum of the roots and the sum of the roots taken two at a time of $g(x) = 1 + \sum_{n=1}^{2012} nx^n$. This is just

$$-\frac{2011}{2012} + \frac{2010}{2012} = -\frac{1}{2012}, \ \boxed{\text{B.}}$$

18. We proceed by casework. Our first case, a real result, can be achieved by rolling both real numbers or both imaginary numbers. Note that both the first and second subcases are symmetric – so the total expected value is

$$2\left(\frac{1}{36}\left[(4+5+6)(1+2+3)\right]\right)=2\times\frac{15\cdot 6}{36}=5.$$

Our second case, an imaginary result, is achieved when we multiply an imaginary number by a real number. The expected value of this is

$$-\frac{1}{36}\left[(1+2+3)(1+2+3)+(4+5+6)(4+5+6)\right]=-\frac{29}{4}.$$

The total expected value is $5-\frac{29}{4}=-\$2.25$, $\boxed{\text{A.}}$

19. Writing the expression in cis form gives us

$$\begin{aligned}\left(1+i\sqrt{3}\right)^{2013}&=\left[2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]^{2013}\\&=2^{2013}\left(\text{cis}\left(\frac{\pi}{3}\right)\right)^{2013}\\&=2^{2013}\text{cis}\left(\frac{2013\pi}{3}\right)\\&=2^{2013}\text{cis}\,\pi\\&=-2^{2013},\ \boxed{\text{A.}}\end{aligned}$$

20. Let the first term be a and a common ratio be r. If, at some point in the series, the nth term in the series equals the first, we have $a=ar^{n-1}\Rightarrow r^k=1,\ k=n-1.$ Thus the possible ratios are the nth roots of unity. There must be 50 of these roots in the second quadrant, or between $90°$ and $180°$. Since the roots of unity are $\text{cis}\left(\frac{2\pi k}{n}\right)$, for some x, we must have

$$\frac{360x}{k}<90\Rightarrow x<\frac{k}{4}$$
$$\frac{360(x+50)}{k}<180\Rightarrow x<\frac{k}{2}-50$$

Subtracting the second from the first gives $\frac{k}{4} > 50 \Rightarrow k > 200.$ Thus the smallest value of k is $k = 201,$ which gives $n = k+1 = 201+1 = 202,$ $\boxed{\text{C.}}$

21. This is just

$$\begin{aligned} f(i) &= 1 - \frac{i^2}{2!} + \frac{i^4}{4!} \\ &= 1 + \frac{1}{2} + \frac{1}{24} \\ &= \frac{37}{24}, \ \boxed{\text{D.}} \end{aligned}$$

22. Note that $\text{cis}\theta_1 \text{cis}\theta_2 = \text{cis}(\theta_1 + \theta_2).$ Using this, we have

$$\begin{aligned} \prod_{\theta=1}^{2013} \text{cis}\theta^\circ &= \text{cis}1^\circ \text{cis}2^\circ \cdots \text{cis}2013^\circ \\ &= \text{cis}\left(\frac{2013(2014)}{2}\right) \\ &= \text{cis}(1007 \cdot 2013)^\circ, \ \boxed{\text{B.}} \end{aligned}$$

23. The plotted points form a spiral shape, composed of segments which we can treat as hypotenuses of right triangles for our purposes of calculating distance. Since the powers of i traverse the axes counterclockwise, each two set of consecutive points along with the origin form a right triangle. For example, $z_1 = \sqrt{\binom{2}{2}} i^1 = i,$ and $z_2 = \sqrt{\binom{3}{2}} i^2 = -\sqrt{3},$ giving $z_1 z_2 = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2.$ In general, we have

$$\begin{aligned} z_k z_{k+1} &= \sqrt{\binom{k+1}{2} + \binom{k+2}{2}} \\ &= \sqrt{\frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}} \\ &= \sqrt{(k+1)^2} \\ &= k+1. \end{aligned}$$

Thus,

$$\begin{aligned} z_1z_2 + z_2z_3 + \cdots + z_{2012}z_{2013} &= 2 + 3 + \cdots + 2013 \\ &= \frac{2013 \cdot 2014}{2} - 1 \\ &= 2027090. \end{aligned}$$

Therefore, our answer is $2027090 \pmod{100} \equiv 90$, $\boxed{\text{C.}}$

24. Note that A is a 60° counterclockwise rotation matrix. So every $\frac{360^\circ}{60^\circ} = 6$ times we apply it, we simply return to the same vector. This means that

$$\begin{aligned} A^{37}z &= A^{6(6)+1}z \\ &= Az \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 4i \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 3 - 4i\sqrt{3} \\ 3\sqrt{3} + 4i \end{pmatrix}, \ \boxed{\text{B.}} \end{aligned}$$

25. We have

$$\begin{aligned} \sum_{n=1}^{k} n(n!) &= \sum_{n=1}^{k} (n+1-1)(n!) \\ &= \sum_{n=1}^{k} \left[(n+1)(n!) - n!\right] \\ &= \sum_{n=1}^{k} \left[(n+1)! - n!\right] \\ &= \left[(k+1)! - k!\right] + \left[k! - (k-1)!\right] + \cdots + \left[2! - 1!\right] \\ &= (k+1)! - 1 \end{aligned}$$

Thus, the sum becomes

$$\sum_{n=1}^{k} n \cdot n! + 1 = (k+1)! - 1 + 1 = (k+1)!,$$

and we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{i^k}{(k+1)!} &= \frac{1}{i}\sum_{k=0}^{\infty}\frac{i^{k+1}}{(k+1)!} \\ &= \frac{1}{i}\left(\sum_{n=1}^{\infty}\frac{i^n}{n!}\right) \\ &= -i\left(e^i-1\right) \\ &= -ie^i+i, \ \boxed{\text{C.}}\end{aligned}$$

26. We can write $f(x)$ as

$$\begin{aligned} f(x) &= x^{2013}+x^{2012}+\cdots+x+1 \\ &= x^{2012}(x+1)+x^{2010}(x+1)+\cdots+x^2(x+1)+(x+1) \\ &= (x+1)(x^{2012}+x^{2010}+\cdots+x^2+1) \\ &= (x+1)\left(x^{2010}(x^2+1)+\cdots+(x^2+1)\right) \\ &= (x+1)(x^2+1)(x^{2010}+x^{2008}+\cdots+1). \end{aligned}$$

Thus, we know that $f(x)$ has $-1, i,$ and $-i$ as roots. Since the powers of i cycle, we are only worried about the powers of i that come out to $1,$ or every fourth power. Note that $R(1)=f(1)=2014,$ by the Remainder theorem. Since we begin at $k=0,$ our answer is $2014\left(\left\lfloor\frac{2013}{4}\right\rfloor+1\right)(\text{mod}\,100)\equiv 1015056(\text{mod}\,100)\equiv 56,$ $\boxed{\text{D.}}$

27. The function in this problem is similar to the function given in Problem 26. We can write $f_n(x)$ as

$$\begin{aligned} f_n(x) &= \sum_{j=0}^{2^n-1} x^j \\ &= \sum_{j=0}^{2^{n-1}-1}\left(x^{2j+1}+x^{2j}\right) \\ &= (1+x)\sum_{j=0}^{2^{n-1}-1} x^{2j} \\ &= (1+x)\sum_{j=0}^{2^{n-2}-1}\left(x^{4j+2}+x^{4j}\right) \\ &= (1+x)(1+x^2)\sum_{j=0}^{2^{n-2}-1} x^{4j} \\ &\vdots \\ &= (1+x)\left(1+x^2\right)\cdots\left(1+x^{2^{n-1}}\right). \end{aligned}$$

Solving $x^{2^{n-1}} = -1$ gives us $x = \text{cis}\left(\frac{\pi}{2^{n-1}} + \frac{\pi k}{2^{n-2}}\right) \Rightarrow \psi_n = \left\{\frac{\pi}{2^{n-1}}, \frac{3\pi}{2^{n-1}}, \cdots, \frac{\left(2^{n-1}-1\right)\pi}{2^{n-1}}\right\}$. Note that the entire set ψ is a union of all ψ_i. The sum for a given ψ_n is

$$\begin{aligned}\frac{\pi}{2^{n-1}}\left(1+3+\cdots+\left(2^{n-1}-1\right)\right) &= \frac{\pi}{2^{n-1}}\left(2^{n-2}\right)^2 \\ &= \frac{2^{2n-4}}{2^{n-1}}\pi \\ &= 2^n\left(\frac{\pi}{8}\right).\end{aligned}$$

Thus the entire sum (while accommodating for $x+1=0 \Rightarrow x = \text{cis}(\pi)$) is

$$\begin{aligned}\pi + \frac{\pi}{8}\left(2^1 + 2^2 + \cdots + 2^n\right) &= \pi + \frac{\pi}{8}\left(\frac{2\left(2^n-1\right)}{1}\right) \\ &= \pi + \frac{\pi}{4}\left(2^n - 1\right).\end{aligned}$$

Finally, we must find n such that

$$\pi + \frac{\pi}{4}\left(2^n-1\right) > 2013\pi \Rightarrow 2^n - 1 > 8048 \Rightarrow n > \log_2 8049.$$

We can easily verify that the smallest such n is 12, $\boxed{\text{C.}}$

28. We know that $|z|^2 = m^2 + 9n^2$. Consider this modulo 8. Since the quadratic residues $\bmod 8$ are $0, 1,$ and $4,$ the possible values of $|z|^2 \pmod 8$ are

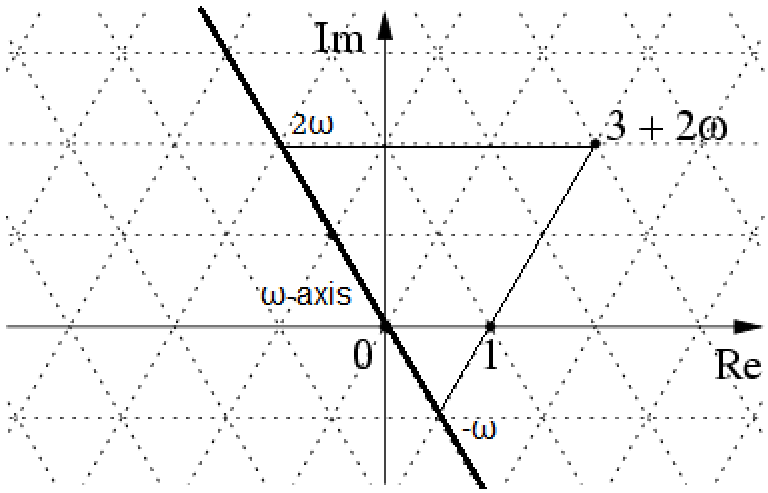
$$\begin{aligned}0+0 &\equiv 0 \pmod 8 \\ 0+1 &\equiv 1 \pmod 8 \\ 0+4 &\equiv 4 \pmod 8 \\ 1+1 &\equiv 2 \pmod 8 \\ 1+4 &\equiv 5 \pmod 8\end{aligned}$$

The answer choices, $\bmod 8,$ are

$$2010 \equiv 2 \pmod 8$$
$$2011 \equiv 3 \pmod 8$$
$$2012 \equiv 4 \pmod 8$$
$$2013 \equiv 5 \pmod 8$$

Thus our answer is 2011, $\boxed{\text{B.}}$

29. Our intuition for a new set of axes is based on the fact that the Eisenstein integers have an argument of $60°$. Through some playing around, we can find the set of axes as shown below:



As we can see, the plotted points form an equilateral triangle with a side length of 3. Thus, the area is $\frac{3^2\sqrt{3}}{4} = \frac{9\sqrt{3}}{4}$, $\boxed{\text{C.}}$

30. We have

$$\begin{aligned} z &= a + b\omega \\ &= a + b\left(\frac{1}{2}(-1 + i\sqrt{3})\right) \\ &= \left(a - \frac{b}{2}\right) + i\left(\frac{b\sqrt{3}}{2}\right). \end{aligned}$$

Thus,

$$\begin{aligned}|z| &= \sqrt{\left(a-\frac{b}{2}\right)^2+\left(\frac{b\sqrt{3}}{2}\right)^2}\\ &= \sqrt{a^2-ab+\frac{b^2}{4}+\frac{3b^2}{4}}\\ &= \sqrt{a^2-ab+b^2}\\ &= \sqrt{(a-b)^2+ab},\ \boxed{\text{C.}}\end{aligned}$$