

Alpha Equations & Inequalities 2013 MAΘ National Convention

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

- Four squares, all of side length 1, are arranged into a single figure such that each square is connected by an edge. What is the positive difference between the minimum and maximum possible perimeters of such figures?
(A) 2 (B) 3 (C) 4 (D) 5 (E) NOTA
- How many integers x satisfy the inequality $|-2|x|+3| < 15$?
(A) 13 (B) 15 (C) 17 (D) 19 (E) NOTA
- If $f(x) = 1/(x + 2)$ and $g(x) = (x + 2)/x$, evaluate $f(g(2))$.
(A) $1/6$ (B) $1/4$ (C) $1/2$ (D) 2 (E) NOTA
- Find the sum of the real solutions x to $x^2 + 12x + 35 = 0$.
(A) -35 (B) $-35/12$ (C) 12 (D) -12 (E) NOTA
- Solve for x : $\log_2(x^2) + \log_2(2x) = 4$
(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA
- Jenny is twice as old as Craig, and Craig is five years older than Lucy. If Jenny is three times as old as Lucy, how old is Jenny, in years?
(A) 23 (B) 25 (C) 27 (D) 29 (E) NOTA
- Let $f(x) = ax^2 + bx + c$ such that $f(0) = f(1)$, $f(2) = 2f(1)$, and $f(3) = 16$. Calculate $f(-1)$.
(A) 4 (B) 6 (C) 8 (D) 9 (E) NOTA
- Farmer Arnold raises chickens, but wants to eat bacon. Farmer Schwartz raises pigs, but wants to eat fried chicken. They want to trade, but they also want to be fair. If they agree that one chicken is worth 7 dollars, and one pig is worth 11 dollars, what is the smallest dollar value of a fair trade?
(A) 33 (B) 49 (C) 66 (D) 91 (E) NOTA

9. (Continuation of Problem #8). Prices have increased and chickens are now worth 14 dollars and a pig is 21 dollars. However, Farmer Schwartz realizes it's a pain to move pigs one at a time, and so he will only trade pigs in sets of 3. What is the smallest dollar value of a fair trade?
- (A) 126 (B) 156 (C) 180 (D) 210 (E) NOTA
10. Given a circle with equation $x^2 + y^2 = 25$, and an ellipse with equation $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$, how many intersection points are there?
- (A) 0 (B) 1 (C) 2 (D) 4 (E) NOTA
11. Let a , b , and c be the solutions of $x^3 - 8x^2 + 10x + 10 = 0$. Find $a + b + c$.
- (A) $4/5$ (B) $-4/5$ (C) -10 (D) 10 (E) NOTA
12. Given a non-degenerate triangle with integer side lengths such that two sides have lengths 14 and 11, what is the longest possible third side length?
- (A) 23 (B) 24 (C) 25 (D) 26 (E) NOTA
13. A shopper was buying scarves and sweaters, 13 in total. If scarves cost 2 dollars less than sweaters, both prices are integers, and the shopper spent 90 dollars altogether, how much do sweaters cost, in dollars?
- (A) 4 (B) 6 (C) 8 (D) 10 (E) NOTA
14. Jon, Rick, and Amy share a lawn. If Jon and Rick can mow the lawn in 4 hours, Rick and Amy can mow the lawn in 5 hours, and Jon and Amy can mow it in 4.5 hours, how many hours will it take the three of them together?
- (A) 2 (B) $48/17$ (C) $144/55$ (D) $360/121$ (E) NOTA
15. Which of the following statements is *false*?
- (A) The sum of two even functions is even.
(B) The sum of two odd functions is odd.
(C) The product of two even functions is even.
(D) The product of two odd functions is odd.
(E) NOTA

16. Solve for x : $\sqrt{x + \sqrt{7 + \sqrt{x + \sqrt{7 + \dots}}}} = 2$
- (A) 1 (B) 3 (C) 5 (D) $\frac{49}{16}$ (E) NOTA
17. Which of the following polynomials has no rational roots?
- (A) $x^2 + 4x + 4$ (B) $x^3 + 7x^2 + 14x + 7$
- (C) $x^4 + 6x^2 - 40$ (D) $x^3 + 3x^2 - 4x - 12$ (E) NOTA
18. If $\sec x + \tan x = \frac{7}{3}$, what is the value of $\sec x - \tan x$?
- (A) $-\frac{4}{5}$ (B) $-\frac{7}{3}$ (C) $\frac{3}{7}$ (D) $\frac{5}{4}$ (E) NOTA
19. Given $\sin(2x + y) = \frac{2}{5}$ and $\sin(2x - y) = \frac{3}{5}$, what is the value of $\sin x \cos x \cos y$?
- (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{18}$ (E) NOTA
20. If I throw a dart at the region bounded by a circle of radius 1 centered at the origin, which of the following inequalities represents a region in which the dart has a $1 - \frac{2}{\pi}$ probability of landing?
- (A) $x + y \geq 0$ (B) $x \leq \frac{1}{2}$ (C) $||x| + |y|| > 1$ (D) $x + 2y > -.2$ (E) NOTA
21. Ron paddles his canoe upstream from his house to the park and arrives in 2 hours. On the way back, the trip takes 1 hour. Assuming the park is 4 miles away, and Ron is paddling at a constant rate, what is the speed of the current, in miles per hour?
- (A) 1 (B) $\frac{2}{3}$ (C) $\frac{1}{3}$ (D) 0 (E) NOTA
22. This time, Ron takes a different river and goes to the zoo. The trip again takes 2 hours upstream, and 1 hour downstream, but he forgot to measure how far away the zoo was. If he knows the zoo was at most 5 miles away, what is the maximum speed of the current, given that Ron was traveling at a constant rate? Express your answer in miles per hour.
- (A) $\frac{1}{3}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{5}{4}$ (E) NOTA

23. John, Jack, Jumpy, and Jot are out to watch a movie, and you are standing in line behind them. John pays for the tickets and then notices that the tickets were not the same price. Jot asks, "Well, how much were they"? You lean in to listen, and you hear John say, "The cost of our tickets multiply to $2^9 \cdot 3^2$, and sum to...", but you can't quite catch the last bit. Jot says, "Well, that's not enough information...", and John says, "Well, yours was the most expensive." You butt in and say, "Jot, i'mma let you finish, but your tickets cost..."

(A) 6,6,8,16 (B) 8,8,8,9 (C) 4,8,9,16 (D) 3,8,8,24 (E) NOTA

24. Assuming x , y , and z are positive real numbers, and given $x^3 + y^3 + z^3 = 4$, what is the maximum value of $(1 + x^3)(1 + y^3)(1 + z^3) - (yz)^3 - (xz)^3 - (xy)^3$?

(A) 101/64 (B) 99/64 (C) 201/27 (D) 199/27 (E) NOTA

25. Given $f(x) - xf(-x) = 5$ for all real numbers x , compute $f(1)$.

(A) 4 (B) 5 (C) 6 (D) 7 (E) NOTA

26. Given $f(n) = f(n + 1)f(n - 1)$, $f(1) = 1$, $f(2) = 2$, compute $\sum_{i=1}^{2013} \log_2 f(i)$.

(A) 1 (B) 2 (C) 2012 (D) 2013 (E) NOTA

27. Given $xy = x + y$, $yz = y + z$, $xz = x + z$, compute $\frac{xyz}{(xy+yz+xz)}$.

(A) 1/6 (B) 1/9 (C) 3/4 (D) 1 (E) NOTA

28. Denote the domain of a function f to be $Dom(f)$. Let f be a function such that if $x \in Dom(f)$, then $\frac{1}{x} \in Dom(f)$. If $f(x)f\left(\frac{3}{x}\right) = x$ for all $x \in Dom(f)$, solve for the product of all values of x that satisfy the above equality.

(A) -9 (B) -3 (C) 3 (D) 9 (E) NOTA

29. How many integer solutions (m, n) exist to the following equation:

$$n^2 + 4n + 2 = (m + 1)(m + 2)(m + 3)(m + 4)?$$

(A) 7 (B) 8 (C) 10 (D) 14 (E) NOTA

30. Given a , b , and c are positive real numbers, what is the minimum value of $a(1 + a^2) + b(1 + b^2) + c(1 + c^2)$, given $abc = 6$?

(A) $12 + \sqrt{2}$

(B) $18 + \sqrt[3]{6}$

(C) 9

(D) $4\sqrt{6}$

(E) NOTA