1. Taking the natural logarithm of both sides yields $(2x+2)\ln 3=(6x+3)\ln 5$ from which we re-arrange and discover that $x=\dfrac{3\ln 5-2\ln 3}{2\ln 3-6\ln 5}$. B
2. The resulting matrix from the product will not have an inverse as long as it's determinant is zero. We first find the product

$$AL=\begin{bmatrix}\log x & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}2\log x & 0 & 1\\ 0 & 1 & 2\end{bmatrix}=\begin{bmatrix}2(\log x)^2 & 1 & \log x+2\\ 0 & 2 & 4\end{bmatrix}.$$

From here it follows that

$$ALI=\begin{bmatrix}2(\log x)^2 & 1 & \log x+2\\ 0 & 2 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}2(\log x)^2+\log x+2 & 1\\ 4 & 2\end{bmatrix}$$

Multiplying by S, the identity matrix, does not change anything. So that $ALISS=\begin{bmatrix}2(\log x)^2+\log x+2 & 1\\ 4 & 2\end{bmatrix}$. Finally, we get

$$ALISSA=\begin{bmatrix}2(\log x)^2+\log x+2 & 1\\ 4 & 2\end{bmatrix}\begin{bmatrix}\log x & 1\\ 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}2(\log x)^3+(\log x)^2+2\log x & 2(\log x)^2+\log x+4\\ 4\log x & 8\end{bmatrix}$$

Taking the determinant gives $8(\log x)^3+4(\log x)^2=0$ from which we see that solutions are $x=1,e^{-\frac{1}{2}}$. Their product gives the answer, A.
3. In order to determine $\sinh^{-1}x$ we must switch the places of x and y and solve. We get $x=\dfrac{e^y-e^{-y}}{2}\rightarrow 2xe^y=e^{2y}-1\rightarrow e^{2y}-2xe^y-1=0$. This is a quadratic in e^y and can be solved with the quadratic formula. We get

$$e^y=\frac{2x\pm\sqrt{(-2x)^2+4}}{2}=x+\sqrt{x^2+1}$$ since any exponential must be positive.

Thus $y=\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)\rightarrow\sinh^{-1}5=\ln(5+\sqrt{26})$. D

4. This is just testing general logarithmic rules.

$$3\log_4 x - 4\log_2 y + \log_8 z = \frac{\log x^3}{\log 4} - \frac{\log y^4}{\log 2} + \frac{\log z}{\log 8} = \frac{\log x^3}{2\log 2} - \frac{\log y^4}{\log 2} + \frac{\log z}{3\log 2} =$$

$$\frac{3\log x^3 - 6\log y^4 + 2\log z}{6\log 2} = \frac{\log x^9 - \log y^{24} + \log z^2}{\log 2^6} = \frac{\log\left(\frac{x^9 z^2}{y^{24}}\right)}{\log 64} = \log_{64}\left(\frac{x^9 z^2}{y^{24}}\right)$$

.A

5. Adding $2xy$ to both sides of the first equation yields $(x+y)^2 = 16xy$ which is equivalent to $\left(\frac{1}{4}(x+y)\right)^2 = xy$. Taking the log of both sides gives

$$2\log\left(\frac{1}{4}(x+y)\right) = \log x + \log y \Rightarrow \log\left(\frac{1}{4}(x+y)\right) = \frac{1}{2}(\log x + \log y)$$. Thus $k = \frac{1}{4}$.

B

6.

$$\ln(\sin(\alpha+\pi)) - \ln(\cos(\alpha+\pi)) = 0 \Rightarrow \ln\left(\frac{\sin(\alpha+\pi)}{\cos(\alpha+\pi)}\right) = 0 \Rightarrow \ln(\tan(\alpha+\pi)) = 0$$

$$\Rightarrow \tan(\alpha+\pi) = 1 \Rightarrow \alpha+\pi = \arctan(1) = \frac{\pi}{4} \Rightarrow \alpha = \frac{\pi}{4} - \pi = \frac{-3\pi}{4} = \frac{5\pi}{4}$$

C

7. We will have horizaontal asymptotes as x approaches plus and minus infinity. As x tends towards negative infinity we get $y=0$ and as x tends towards positive infinity we get $y=2$. Hence the sum is 2, B.

8. $2013^{1000} = 10^n \Rightarrow n = 1000\log 2013 \approx 3303.84$. Round down and add 1 to determine number of digits. 3304 B

9. Any half-life problem can be reduced to the following equation $A = P\left(\frac{1}{2}\right)^t$ where t is time and P is the principle or starting amount. The number of subsets of a set with cardinality n is 2^n , hence there are $2^{10} = 1024$ subsets of $\{1,2,...,10\}$. Using this information we can write

$$2^{-2^8} = P\left(\frac{1}{2}\right)^{2^{10}} \Rightarrow P = 2^{2^{10}-2^8} = 2^{768}$$. C

10. To evaluate infinite nested exponents we use a simple trick. Let

$$y = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}} = \sqrt{2}^y = 2^{\frac{y}{2}}$$. Squaring both sides gives $y^2 = 2^y$ from which it is clear that y=2. A

11. $\prod_{n=1}^{2013}(x-n)=2 \Rightarrow (x-1)(x-2)\ldots(x-2013)=2$. The left hand side is a product of 2013 factors, which gives a 2013 degree polynomial. Taking the constant to the other side does not change this. By the Fundamental Theorem of Algebra this polynomial has 2013 roots (solutions). D

12. $\tan\left(arc\cos\left(-\frac{3}{5}\right)\right)=-\frac{4}{3}$ by using a simple right triangle relation. Thus we have that $e^{-\frac{4}{3}}=x^e \Rightarrow \ln e^{-\frac{4}{3}}=e\ln x \Rightarrow -\frac{4}{3e}=\ln x \Rightarrow x=e^{-\frac{4}{3e}}$

13. We need to solve for both $\sin x$ and $\cos x$. This can be done using the given formula such that $\cos x=\frac{e^{ix}+e^{-ix}}{2}$ and $\sin x=\frac{e^{ix}-e^{-ix}}{2i}$ from which we get

$$\tan x=\frac{\sin x}{\cos x}=\frac{\frac{e^{ix}-e^{-ix}}{2i}}{\frac{e^{ix}+e^{-ix}}{2}}=\frac{e^{ix}-e^{-ix}}{i\left(e^{ix}+e^{-ix}\right)}=\frac{-i\left(e^{ix}-e^{-ix}\right)}{e^{ix}+e^{-ix}}$$ .A

14. $\left(\frac{x^{-1}+y^{-1}+z^{-1}}{xz+yz+yx}\right)^2=\left(\frac{\frac{xz+yz+yx}{xyz}}{x+y+z}\right)^2=\frac{1}{(xyz)^2}$ .B

15. Long division yields $\frac{3e^{3x}-e^{2x}-e^x-4}{e^x+1}=3e^{2x}-4e^x-4=\left(e^x-2\right)\left(3e^x+2\right)$ from which it follows that $x=\ln 2$. B

16. The first equation yields $x^2+y^2=\frac{5}{2}xy$ after rearrangement. Similarly, the second equation can be reworked to give $x^2-y^2=3$. Now, $x^2+y^2=\frac{5}{2}xy \Rightarrow x^2-2xy+y^2=\frac{1}{2}xy \Rightarrow (x-y)^2=\frac{1}{2}xy$. Dividing this by the square of the second yields, $\frac{1}{(x+y)^2}=\frac{xy}{18} \Rightarrow (x+y)^2=\frac{18}{xy}$. We get the system $x^2+2xy+y^2=\frac{18}{xy}$ and $x^2-2xy+y^2=\frac{xy}{2}$, and subtracting yields $4xy=\frac{18}{xy}-\frac{xy}{2} \Rightarrow \frac{9}{2}(xy)^2=18 \Rightarrow (xy)^2=4 \Rightarrow xy=\pm 2$. However, since $(x-y)^2=\frac{1}{2}xy$ it follows that the product must be positive, hence $xy=2$. Substituting back into the system yields a new system, namely $x^2+y^2=5$

and $x^2 - y^2 = 3$. Solving this system yields two possible solutions, $(2,1)$ and $(-2,-1)$. Upon inspection only $(2,1)$ is valid. Thus x =2. B

17. Using the fact that $e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$ we have that $\ln i = \ln e^{\frac{i\pi}{2}} = \frac{i\pi}{2}$. B
18. To determine the number of zeros at the end of a factorial, divide the number by powers of 5 and add the quotients. Thus we have $x = \left\lfloor \frac{2013}{5} \right\rfloor + \left\lfloor \frac{2013}{25} \right\rfloor + \left\lfloor \frac{2013}{125} \right\rfloor + \left\lfloor \frac{2013}{625} \right\rfloor = 402 + 80 + 16 + 3 = 501$. Thus $\log_{167}\left(\frac{x}{3}\right) = \log_{167}\left(\frac{501}{3}\right) = \log_{167}(167) = 1$.B
19. Factoring we have $(1+i)^7(3-3i)^6 = 3^6(1+i)\left((1+i)(1-i)\right)^6 = 6^6(1+i)$ B
20. We will work out this sum from inside to out. So dealing with the inside, it is common knowledge that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. So then we have $\sum_{n=1}^{2012} \ln\left(\frac{n(n+1)}{2}\right)$. Since the sum of logarithms becomes the logarithm of a product we can reduce this as follows: the denominator will consist of a product of 2012 2's, the numerator will consist of the product $1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4\cdots 2012\cdot 2012\cdot 2013 = (2012!)(2013!)$. Thus the result should be $\ln\frac{(2012!)(2013!)}{2^{2012}}$ A.

21. Since $e^{xy} = e^{3x-6x^2} = \frac{e^{3x}}{e^{6x^2}}$ is decreasing to zero as x increases towards positive and negative infinity, we have that this value will be maximized where the exponent is maximized, i.e. where $y = 3x - 6x^2$ is maximized. This is a downward facing parabola, as such its maximum occurs at its vertex $x = 1$. In which case $y = 3$ and $x + y = 4$. D
22. We must first find a common base for both sides: $128^{-3x} = 4096 \Rightarrow 2^{-21x} = 2^{12} \Rightarrow x = -\frac{12}{21}$. B
23. Vertical asymptotes will occur where the argument is zero. Thus at x=5 and x=-2. But since the x+2 term in the denominator cancels this is just a hole. Thus the only vertical asymptote is x=5. C
24. The graph of arctanx has horizontal asymptotes at $y = \pm\frac{\pi}{2}$ thus this graph has horizontal asymptotes at $y = e^{\pm\frac{\pi}{4}}$. C
25. The domain of this function must be so that $\frac{|x+1|}{x^2+3x+2} > 0$ and

$x^2+3x+2\neq 0$. The first inequality is satisfied as long as $x\notin(-2,-1)$. The second inequality is satisfied as long as $x\neq -2,-1$. Thus the domain is $(-\infty,-2)\cup(-1,\infty)$.B

26. Continuously compounded interest is modeled by the formula $A=Pe^{rt}$. In this case A = 1 and we have \$64 after 9 years. Hence we get $64=1e^{9r}\Rightarrow r=\frac{6\ln 2}{9}=\frac{2\ln 2}{3}$. Thus to determine the amount after 6 years we use the same formula and get $A=1e^{6\left(\frac{2\ln 2}{3}\right)}=e^{4\ln 2}=16$. D

27. We can use a counting argument to determine the binomial coefficients. In the expansion we are looking for terms that have one 'x', one 'y', and four 'z'. As such we can first choose the four 'z' from the six factors, in $\binom{6}{4}=15$ ways. We can choose a 'y' from the remaining two factors in exactly $\binom{2}{1}=2$ ways. The remaining factor will give us our 'x' in one way. Thus the term will be $(15)(2)(1)(2z)^4(3y)(x)=1440xyz^4$. C

28. $\frac{f(x+h)-f(x)}{h}=\frac{\ln(x+h)+e^{x+h}-\ln x-e^x}{h}=\frac{\ln\left(1+\frac{h}{x}\right)+e^x\left(e^h-1\right)}{h}$. B

29. We need to change bases in the following manner:

$$\log_5 x+\log_{25}2x+\log_{125}3x=625\Rightarrow\frac{\log x}{\log 5}+\frac{\log 2x}{\log 5^2}+\frac{\log 3x}{\log 5^3}=625\Rightarrow\frac{\log x}{\log 5}+\frac{\log 2x}{2\log 5}+\frac{\log 3x}{3\log 5}=625=$$

$$\frac{6\log x+3\log 2x+2\log 3x}{6\log 5}=625\Rightarrow\log\left(72x^{11}\right)=3570\log 5\Rightarrow 5^{3570}=72x^{11}\Rightarrow x=\sqrt[11]{\frac{5^{3570}}{72}}$$

C

30. This is an infinite geometric sum with common ratio $\frac{1}{2}$. Note $\log e+\log\sqrt{e}+\log\sqrt[4]{e}+\log\sqrt[8]{e}+\ldots=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. Thus the sum is $\frac{1}{1-\frac{1}{2}}=2$

A.

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11 **(C)**. Either the base equals 1, in which case, $x = 1$; the exponent equals 0 and the base is nonzero, in which case $x \in \{2, 3\}$; or the base is -1 and the exponent is even, in which case $x = -1$. The sum of the fourth powers of all solutions is $(-1)^4 + 1^4 + 2^4 + 3^4 = 99$.

14 **(B)**. The expression $\log_9 n$ will only be rational for integer n if n is a power of 3. On the interval $[1, 2013]$, there are only seven powers of 3: $3^0, 3^1, \ldots, 3^6$. For these values, the sum of the outputs from the function is $0 + .5 + 1 + 1.5 + 2 + 2.5 + 3 = 10.5$. For the other $2013 - 7 = 2006$ values, the sum of the outputs of the function is $2006(.10) = 200.60$, making the total sum equal $10.5 + 200.60 = 211.10$.

19 **(B)**. Change all logarithms to base 4. For the first equation, this becomes

$$\frac{\log_4 x}{3/2} + \log_4 y^2 = \frac{2\log_4 x}{3} + \log_4 y^2 = \log_4(x^{2/3}y^2) = 5$$

For the second equation, this becomes

$$\frac{\log_4 y}{3/2} + \log_4 x^2 = \frac{2\log_4 y}{3} + \log_4 x^2 = \log_4(y^{2/3}y^2) = 7$$

Add the two equations together and use some log rules to obtain $\log_4(xy)^{8/3} = 12$, or $4^{12} = (xy)^{8/3}$, or $xy = 2^9 = 512$.

22 **(A)**. If $(\log_3 p)^2 = \log_3 p^2$, then either $\log_3 p = 0$ or $\log_3 p = 2$, leading to $p \in \{1, 9\}$. The second equation simplifies to $\log_3(p + q) = \log_3(pq)$, or $p + q = pq$. If $p = 1$, then this leads to the contradiction $1 + q = q$. Thus, $p = 9$ and $q = p/(p - 1) = 9/8$.

26 **(D)**. We have

$$x^{(\log_2 7)^2} + y^{(\log_3 5)^2} + z^{(\log_5 216)^2} = \left(x^{(\log_2 7)}\right)^{(\log_2 7)} + \left(y^{(\log_3 5)}\right)^{(\log_3 5)} + \left(z^{(\log_5 216)}\right)^{(\log_5 216)}$$

or

$$\left(x^{(\log_2 7)}\right)^{(\log_2 7)} + \left(y^{(\log_3 5)}\right)^{(\log_3 5)} + \left(z^{(\log_5 216)}\right)^{(\log_5 216)} = (8)^{(\log_2 7)} + (81)^{(\log_3 5)} + \left(\sqrt[3]{5}\right)^{(\log_5 216)}$$

or

$$(2)^{(\log_2 7^3)} + (3)^{(\log_3 5^4)} + (5)^{(\log_5 216^{1/3})} = 7^3 + 5^4 + 216^{1/3} = 974.$$

28 **(D)**. The given information can be re-written as $2\log 7 = a$ and $4\log 5 = b$. The second equation can be manipulated to yield

$$\log 5 = \log(10/2) = b/4 \to \log 10 - \log 2 = \frac{b}{4} \to 1 - b/4 = \log 2 \to 2 - b/2 = \log 4$$

Thus, we have

$$\log\frac{1}{28} = -(\log 7 + \log 4) = -\left(\frac{a}{2} + 2 - \frac{b}{2}\right) = \frac{b - a}{2} - 2.$$

29 **(C)**. Let $u = \log_{4096} x$ and $v = \log_{2013} y$ so that the equations become $u + v = 2$ and $1/u - 1/v = 1$; this system has solutions $(u, v) = \left(2 \mp \sqrt{2}, \pm\sqrt{2}\right)$, so that $u_1 + u_2 = \log_{4096} x_1 + \log_{4096} x_2 = \log_{4096}(x_1 x_2) = 4$ and $v_1 + v_2 = \log_{2013} y_1 + \log_{2013} y_2 = \log_{2013}(y_1 y_2) = 0$. By change-of-base, if $\log_{4096}(x_1 x_2) = 4$, then $\log_4(x_1 x_2) = 24$ and if $\log_{2013}(y_1 y_2) = 0$, then $y_1 y_2 = 2013^0 = 1$. Thus, $\log_4(x_1 x_2 y_1 y_2) = 24$.