Mu Integration
2013 Mu Alpha Theta National Convention
Solutions

1. Rewrite $\frac{-x^2}{1+x^2}$ as $\frac{-1-x^2}{1+x^2}+\frac{1}{1+x^2}=-1+\frac{1}{1+x^2}$ so that our integral becomes $\int\left(-1+\frac{1}{1+x^2}\right)dx=-x+\arctan x+C$. B
2. This is testing a version of the fundamental theorem of calculus. Taking the derivative of the integral yields $f'(x)=\frac{1}{x\ln x}\to f'\left(\frac{1}{e}\right)=-e$. D
3. This is a Riemann Sum. Dividing and re-arranging gives $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4n}{n^2+i^2}=4\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}\frac{1}{n}=4\int_0^1\frac{dx}{1+x^2}=4\arctan 1-4\arctan 0=\pi$. B
4. Using the substitution $u=\frac{\sin x}{2}$ turns the integral into $2\int_0^{\infty}e^{-u^2}\,du=\int_{-\infty}^{\infty}e^{-u^2}\,du=\sqrt{\pi}$.
5. Since n>-1 the integral is well defined using integration by parts. Thus we let $dv=x^n\Rightarrow v=\frac{x^{n+1}}{n+1}$ and $u=\ln x\Rightarrow du=\frac{dx}{x}$ so that $\int_1^e x^n\ln x\,dx=\frac{(\ln x)x^{n+1}}{n+1}-\frac{1}{n+1}\int_1^e x^n\,dx=\frac{(\ln x)x^{n+1}}{n+1}-\frac{x^{n+1}}{(n+1)^2}\Big|_1^e=\frac{e^{n+1}}{n+1}-\frac{e^{n+1}-1}{(n+1)^2}=\frac{ne^{n+1}+1}{(n+1)^2}$

   A
6. $\frac{d\left(\int_x^y dx\right)}{dy}=x\Rightarrow\frac{d(y-x)}{dy}=x\Rightarrow 1-\frac{dx}{dy}=x\Rightarrow\frac{1}{1-x}=\frac{dy}{dx}\Rightarrow\int\frac{dx}{1-x}=\int dy\Rightarrow y=-\ln(1-x)+C$

   B
7. Let $x=u^3$ then the integral becomes $\int\frac{3du}{u(u+1)}=3\int\frac{du}{u}-3\int\frac{du}{1+u}=3\ln(u)-3\ln(u+1)+C=3\ln\left(x^{\frac{1}{3}}\right)-3\ln\left(x^{\frac{1}{3}}+1\right)+C=$ $\ln|x|-3\ln\left|x^{\frac{1}{3}}+1\right|+C$

   D

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8. Factoring constants out of the radical reveals an easier approach: $2\int_{-a}^{a}\frac{\sqrt{a^2b^2-x^2b^2}}{a}dx = 2\int_{-a}^{a}|b|\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx$ and we note that $y=|b|\sqrt{1-\left(\frac{x}{a}\right)^2}$ is the top half of an ellipse from –a to a, thus the integral represents the area of an ellipse with major radius a and minor radius b which is just $|ab\pi|$.
9. The first part of the problem is to recognize that $\sin x = \int \cos x\,dx = \int \sum_{i=0}^{\infty}\frac{(-1)^i x^{2i}}{(2i)!}dx = \sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+1}}{(2i+1)!}$. Thus $\int_0^{\frac{3\pi}{2}}\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+1}}{(2i+1)!}dx = \int_0^{\frac{3\pi}{2}}\sin x\,dx = -\cos\frac{3\pi}{2}+\cos 0 = 1$.
10. Given $a(t)=2 \Rightarrow v(t)=2t-\sqrt{10}(2000000) \Rightarrow s(t)=t^2-\sqrt{10}(2000000)t$. Thus we need the time such that $t^2-\sqrt{10}(2000000)t+10000000000000=0$. Note that the constant is positive because the particle is traveling downward and starts at position zero, thus we start with $t^2-\sqrt{10}(2000000)t=-10000000000000$ Using the quadratic formula is convenient here because the discriminant is zero. Thus the answer is $\sqrt{10}(1000000)$. C
11. ~~The region formed by the intersection of these graphs is a triangle with base length 2 and height 4/3 (which occurs at x=1/3 , the intersection of the two lines). Thus the area is (1/2)(2)(4/3)=4/3. D~~
12. Let $u=x^2-1$ then the integral becomes $\frac{1}{2}\int \arctan u\,du = \frac{1}{2}\left(u\arctan u-\frac{1}{2}\ln\left|(u^2+1)\right|\right)+C =$ $\frac{1}{2}(x^2-1)\arctan(x^2-1)-\frac{1}{4}\ln\left|(x^4-2x^2+2)\right|+C$ through integration by parts. Evaluating at the bounds gives $\frac{\pi}{8}-\frac{1}{4}\ln 2$ .B
13. Attempting to do this integral is futile since the bounds cross regions where $y=\sec^5 x$ has vertical asymptotes. Thus the integral does not exist, NOTA. E
14. Average value = $\frac{1}{2\pi}\int_0^{2\pi}\sin x dx = \frac{1}{2\pi}(-\cos 2\pi+\cos 0)=0$.

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15. This integral uses a clever trick involving $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot x}$. Note that on the interval of integration both tanx and cotx map out the same area. Thus $I=\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}=\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot x}$ and $2I=\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}+\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot x}=\int_0^{\frac{\pi}{2}} dx=\frac{\pi}{2} \Rightarrow I=\frac{\pi}{4}$. B
16. The volume of a solid with known cross sections (perpendicular to the y axis) with area A is $\int_{-b}^{b} A\,dy$. The cross sections are squares whose base has length $2\sqrt{\frac{36-9y^2}{4}}$. Thus $A=\left(2\sqrt{\frac{36-9y^2}{4}}\right)^2=36-9y^2$. To determine the volume we evaluate $\int_{-2}^{2}\left(36-9y^2\right)dy=36x-3y^3\Big|_{-2}^{2}=96$.D
17. ~~The region is a rectangular prism with one corner at the origin and sides with length 2, 5, and 10. Thus the volume is (2)(5)(10)=100. C~~
18. The function has zeros at x = 2 and x =6. Simpson's Rule for quadratics gives the exact area under the curve and thus we can just determine the definite integral which evaluates to 32/3.

    B

19. i) False

    ii) True

    iii) True since the left side is a left hand approximation and the right side is a right side approximation. Draw a picture to help visualize.

20. The solid generated is a torus and you can use geometry to determine its volume. Imagine the volume of the torus as being generated by slices of circles with area π (the area of $(x-5)^2+y^2=1$). There are continuous circle slices equivalent to the circumference of the revolved region. Note that the centroid of the circle moves through a circular path which has a radius of 5 (distance between the y axis and center of $(x-5)^2+y^2=1$). Therefore the volume is the area of our circle, π, multiplied by the circumference of our revolving circle, 10π. The volume is thus $10\pi^2$. C

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21. First we must rewrite our infinite function. This is done as follows:

$$y=\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\ldots}}}=\frac{x}{1+y}\Rightarrow y^2+y=x\Rightarrow y^2+y+\frac{1}{4}=x+\frac{1}{4}\Rightarrow\left(y+\frac{1}{2}\right)^2=x+\frac{1}{4}\Rightarrow$$

$$y=\sqrt{x+\frac{1}{4}}-\frac{1}{2}$$

Thus the integral becomes $\int_0^{\frac{3}{4}}\left(\sqrt{x+\frac{1}{4}}-\frac{1}{2}\right)dx=\frac{5}{24}$.B

22. $\int_1^4\left(x^2+2\right)dx=\frac{x^3}{3}+2x\Big|_1^4=\frac{64}{3}+8-\frac{1}{3}-2=27$ A

23. The region is a triangle and thus we can use geometry to determine the integral. The triangle has a base length of 8 and a height of 8. Thus the area is (1/2)(8)(8)=32. D

24. Polar area is defined as $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$ thus all we need to do is determine the angles which map out one petal of the graph. These are going to be $\theta=-\frac{\pi}{12}$ and $\theta=\frac{\pi}{12}$ which can be found by sketching and looking for where the graph returns to the origin, i.e. where r=0. Thus the area is

$$\frac{1}{2}\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}}4\cos^2(6\theta)d\theta=4\int_0^{\frac{\pi}{12}}\frac{\cos(12\theta)+1}{2}d\theta=\frac{\sin(12\theta)}{6}+2\theta\Big|_0^{\frac{\pi}{12}}=\frac{\pi}{6}.\text{B}$$

25. Arclength = $\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,ds$. Thus we have that arclength = $\int_0^4\sqrt{\sin^2x+\cos^2x}\,ds=4$. D

26. Since f is even we have that $\int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx=12$ and since g is odd we have that $\int_{-a}^{a}g(x)=0$. Thus $\int_{-a}^{a}\left(2f(x)-3g(x)\right)dx$=24. D

27. $\int\left(x^2+x-3\right)dx=\frac{x^3}{3}+\frac{x^2}{2}-3x+C$. E

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28. ~~This integral represents the area of a semi-circle with radius $\sqrt{2013}$. Thus $\int_{-\sqrt{2013}}^{\sqrt{2013}} \sqrt{2013-x^2}\,dx = \frac{2013\pi}{2}$. C~~
29. Using u-sub, let $u = x+1$ thus the integral becomes $\int x\sqrt{x+1}\,dx = \int (u-1)u^{\frac{1}{2}}du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$.B
30. Fundamental Theorem of Calculus in addition to second derivatives. Thus we have $\frac{d^2}{dx^2}\left(\int_1^{x^2} \ln\sec t\,dt\right) = \frac{d}{dx}\left(2x\ln\sec\left(x^2\right)\right) = 4x^2\tan\left(x^2\right) + 2\ln\sec\left(x^2\right)$. A

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11 **(C)**. Notice that

$$\cot^{-1}(1-x+x^2) = \tan^{-1}\frac{1}{1-x+x^2} = \tan^{-1}\frac{x-(x-1)}{1+x(x-1)} = \tan^{-1}x - \tan^{-1}(x-1).$$

Also, notice that

$$\int_0^1 \tan^{-1}(x-1)\,dx = -\int_0^1 \tan^{-1}x\,dx$$

because we can make a substitution $u = x-1$ on the left-hand-side integral to obtain the right-hand-side integral. Therefore,

$$\int_0^1 \cot^{-1}(1-x+x^2)\,dx = \int_0^1 \left(\tan^{-1}x - \tan^{-1}(x-1)\right)\,dx = 2\int_0^1 \tan^{-1}x\,dx,$$

which means that the answer to the problem is equal to 2.

17 **(C)**. Let $u = (x-1)/(b-1)$, so that the integral is transformed to

$$\lim_{b\to 1^+}\int_0^b \frac{1}{\sqrt{x(x-1)(b-x)}}\,dx = \lim_{b\to 1^+}\int_0^1 \frac{1}{\sqrt{u(1-u)(1+(b-1)u)}}\,du.$$

As $b \to 1^+$, clearly we get $\int_0^1 \frac{1}{\sqrt{u(1-u)}}\,du = \pi$.

22 **(A)**. Let I equal the desired integral. Let $x = (1-u)/(1+u)$ to obtain

$$I = \int_0^1 \frac{\ln(x+1)}{1+x^2}\,dx = \int_0^1 \frac{\ln 2 - \ln(1+t)}{1+t^2}\,dt = \int_0^1 \frac{\ln 2}{1+t^2}\,dt - I,$$

so that $2I = \int_0^1 \frac{\ln 2}{1+t^2}\,dt = (\ln 2)(\tan^{-1}1) = (\ln 2)(\frac{\pi}{4})$. Thus, $I = \frac{\pi}{8}\ln 2$.

23 **(D)**. Use Integration by Parts, with $u = x$ and $dv = \frac{\sin x}{1+(\cos x)^2}$. Thus, we have $du = dx$ and $v = -\tan^{-1}(\cos x)$:

$$uv - \int v du = -x\tan^{-1}(\cos x)\Big|_0^\pi + \int_0^\pi \tan^{-1}(\cos x)\,dx$$

The integral on the right-hand-side is equal to 0 because of the symmetry of the arctangent function on the given interval. Thus, the value of the integral is $I = \pi^2/4$, so that $\sin\sqrt{I} = 1$.

25 **(D)**. Using the fact that

$$1+2\cos x+2\cos(2x)+\cdots+2\cos(nx)=\frac{\sin\left((n+1/2)x\right)}{\sin(x/2)}=\sin(nx)\cot(x/2)+\cos(nx),$$

we have

$$a_n=\int_0^{\pi}\cot(x/2)\sin(nx)\,dx=\int_0^{\pi}(1+2\cos x+2\cos(2x)+\cdots+\cos(nx))\,dx=\int_0^{\pi}1\,dx+0=\pi.$$

Thus, $S=2013\pi$ and $\cos S=\cos(2013\pi)=-1$.

27 **(E)**. Notice that, by symmetry,

$$I=\int_0^{\pi}\log_2(\sin x)\,dx=2\int_0^{\pi/2}\log_2(\sin x)\,dx.$$

Also, by symmetry $\int_0^{\pi/2}\log_2(\sin x)\,dx=\int_0^{\pi/2}\log_2(\cos x)\,dx$. Thus, we have:

$$2I=2\left(\int_0^{\pi/2}\log_2(\sin x)\,dx+\int_0^{\pi/2}\log_2(\cos x)\,dx\right)=2\int_0^{\pi/2}\log_2(\sin x\cos x)\,dx$$

or

$$I=\int_0^{\pi/2}\log_2\left(\frac{1}{2}\sin(2x)\right)\,dx=\int_0^{\pi/2}\log_2\frac{1}{2}\,dx+\int_0^{\pi/2}\log_2(\sin(2x))\,dx=-\frac{\pi}{2}+\frac{I}{2}.$$

Thus, $I=-\frac{\pi}{2}+\frac{I}{2}$, or $I=-\pi$, making $\cos I=\cos(-\pi)=\cos(\pi)=-1$.

28 **(C)**. Let $I=\int_0^1 f(x)x^{2812}\,dx$. By Cauchy-Schwarz,

$$I^2\le\left(\int_0^1(f(x))^2\,dx\right)\left(\int_0^1(x^{2812})^2\,dx\right)=(1)\left(\int_0^1 x^{5624}\,dx\right)=\frac{1}{5625}.$$

Thus, $I^2\le\sqrt{1/5625}$, and since I is nonnegative, $I\le 1/75$, so that $m/n=1/75$ and $m+n=1+75=76$.