Mu Limits and Derivatives – Solution Guide

1. **D:** The function is not defined at 3, because of division by zero.
2. **B:** The concept of limits is the only thing that is associated with all three things.
3. **B:** This can be evaluated using L'Hopital's Rule.
4. **E:** This is a repeated application of the Chain Rule. A) would be the right choice, but it off by a factor of -1.
5. **C:** This simply requires plugging in zero for x. Applying L'Hopital's rule would lead to the wrong answer.
6. **D:** The minimum value of f' is -4, since x^2 cannot be negative.
7. ~~**A:** Simple definition question.~~
8. **B:** This is the right end of the domain for the function, so the right-sided limit does not exist.
9. **D:** This is a repeated application of the Product Rule, and then you plug in pi/2.
10. **D:** This involves the Chain Rule. Also, f and g have the same derivative because they differ by a constant. g'(f(1))*f'(1) = g'(2)*f'(1) = 7*4
11. **A:** When you factor out (x-2) from the top and the bottom of the fraction, you get a new fraction: (x+7)(x-4)/(x+6).
12. **E:** It approaches 3/2, because the sqrt(4x^2+1) grows at the same rate as 2x.
13. **D:** The function constantly jumps back and forth between 0 and 1.
14. ~~**B:** You multiply the top and bottom by the conjugate of the numerator: sqrt(x+1)+2. Then things cancel out and simplify.~~
15. ~~**B:** You apply the quotient rule, and then factor out an (x^2+4)^(-1/2) on the top, which will help with the simplification to the answer given.~~
16. **C:** You set up two equations, checking for continuity and differentiability at the point x=2. You get sin(2pi) = 4a + b, and pi*cos(2pi) = 4a. Solving for the two variables, you get a=pi/4 and b=-pi.
17. **D:** When you take the third derivative of f, you get 4*e^(2x-5). Since this is an always-increasing function, the maximum value will be at the right endpoint of the interval (x=3).
18. **C:** The function is only non-differentiable where the quadratic in the absolute value is equal to zero – which occurs at x=2 and x=3.
19. ~~**C:** When you take the natural log of the equation, you are now looking at the limit of ln(c)/(c-1), which gives you zero/zero when you look at c=1. So you can now apply L'Hopital's Rule, to get 1/c, which goes to 1. But this is equal to the limit of the natural log, so the actual limit goes to e^1.~~
20. **A:** When you evaluate the equation using Implicit Differentiation, you get dy/dx = -(6x^2+3y)/(2y+3x). A vertical tangent occurs when the derivative does not exist, so you set the denominator equal to zero. Plugging this into the original equation, you get that the two points where there are vertical tangents are (0,0) and (9/8, -27/16). However, there is also a horizontal tangent at (0,0) – the numerator of the dy/dx equation is also equal to zero – so the answer has to be the other point.
21. **C:** The trick to this problem is applying the properties of logarithms before trying to do any differentiation. It simplifies to ln(x^4+1) – 2ln(sin(4x)+2). Now take the derivative and apply chain rule appropriately.
22. **B:** Both are equivalent to 1+2x+3x^2+4x^3…

23. ~~**E:** This would work as a one-sided limit problem, because the square root prevents x from approaching 2 from both sides.~~
24. **D:** Bernoulli's Inequality is useful, as shown here: http://www.milefoot.com/math/calculus/limits/LimitDefinitionOfE10.htm
25. **D:** b_n – a_n = (1+1/n)^(n+1) – (1+1/n)^n = ((1+1/n)^n)(1+1/n-1) = 1/n*a_n
26. **C:** Because b_n>a_n for all n, 0<b_n – a_n. Also, we know that b_n – a_n < 4/n, which goes to zero as n gets large. This is an application of the Sandwich Theorem, which tells us that b_n – a_n must go to zero as well.
27. **A:** This is the same as asking for the integral of sin(pi*x) from 0 to 1, which is 2/pi
28. **B:** (2.05)^2 = 4.2025, while (1.95)^2 = 3.8025, so 0.1975 is the maximum possible value for delta that will work.
29. **B:** The tangent function has points of discontinuity at odd multiples of pi/2. (2pi)^2 = 39.478, and the largest odd multiple of pi/2 smaller than that is 25pi/2, which is about 39.2699 (this can easily be estimated using pi=3.14). There are 13 odd multiple of pi/2 between pi/2 and 25pi/2.
30. ~~**A:** This is an application of the Fundamental Theorem of Calculus, including the Chain Rule.~~

Mu Alpha Theta National Convention: San Diego, 2013
Mu Limits & Derivatives Test - Updated Solutions

7 **(A)**. Let $f(x) = \tan x - x$ and $g(x) = x^3$. By the Extended Mean Value Theorem, there exists $c \in (0, x)$ such that

$$\frac{(\tan x - x) - (\tan 0 - 0)}{x^3 - 0^3} = \frac{f'(c)}{g'(c)} = \frac{\sec^3 c - 1}{3c^2} = \frac{\tan^2 c}{3c^2} = \frac{1}{3}\left(\frac{\tan c}{c}\right)^2$$

The quantity being squared has a limit of 1 as c approaches 0 (because x approaches 0), via L'Hopital's Rule. Thus, the limit is equal to 1/3.

14 **(B)**. Use the fact that

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \frac{1}{2}.$$

We can write the limit in the problem as

$$\lim_{x\to 0} \frac{1 - \cos(1 - \cos x)}{x^4} = \lim_{x\to 0} \frac{1 - \cos(1 - \cos x)}{(1 - \cos x)^2} \frac{(1 - \cos x)^2}{x^4}$$

or

$$\left(\lim_{x\to 0} \frac{1 - \cos(1 - \cos x)}{(1 - \cos x)^2}\right) \lim_{x\to 0} \left(\frac{1 - \cos x}{x^2}\right)^2 = \lim_{u\to 0} \frac{(1 - \cos u)}{u^2} \lim_{x\to 0} \left(\frac{1 - \cos x}{x^2}\right)^2.$$

Thus, the limit is equal to $(1/2)(1/2)^2 = 1/8$, making $m^2 + n^2 = 1^2 + 8^2 = 65$, which has a digital sum of 11.

15 **(B)**. Via Stirling's Approximation, if n is large $n! \approx n^n e^{-n}\sqrt{2\pi n}$, or

$$\frac{n^n}{n!} \approx \frac{e^n}{\sqrt{2\pi n}} \to \sqrt[n]{\frac{n^n}{n!}} \approx \frac{e}{\sqrt[n/2]{2\pi n}}$$

As $n \to \infty$, the denominator approaches 1; therefore, $L = e$ and $1000L \approx 2718.28$, so the greatest integer less than $1000L$ is 2718; the digital sum is 18.

19 **(C)**. If u, v, and x are small, we know that $e^u \approx 1 + u$, $\ln(1 + v) \approx v - v^2/2$, and $\tan x \approx x + x^3/3$. Thus,

$$\ln(1+\tan x)^{1/x} = \frac{1}{x}\ln(1+\tan x) \approx \frac{1}{x}\ln\left(1 + x + \frac{x^3}{3}\right) \approx \frac{1}{x}\left(x + \frac{x^3}{3} - \frac{1}{2}\left(x + \frac{x^3}{3}\right)^2\right)$$

or $\ln(1 + \tan x)^{1/x} \approx 1 - x/2$. Thus,

$$(1 + \tan x)^{1/x} \approx e^{1-x/2} = e^1 e^{-x/2} \approx e\left(1 - \frac{x}{2}\right) = e - \frac{ex}{2}.$$

Thus, the limit is equal to

$$\lim_{x\to 0} \frac{(1 + \tan x)^{1/x} - e}{x} = \lim_{x\to 0} \frac{e - \frac{ex}{2} - e}{x} = -\frac{e}{2}.$$

23 **(E)**. We know that

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}$$

so that

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{x/2}\,dx = \int_0^\infty \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n e^{x/2}\,dx = \int_0^\infty e^{-x}e^{x/2}\,dx = \int_0^\infty e^{-x/2}\,dx = 2.$$

30 **(A)**. By inspection, notice that $f(x) = 1/(1-x)$ is a solution. Thus, $f'(x) = 1/(1-x)^2$, or $f'(2012/2013) = 1/(1/2013)^2 = 2013^2 = 4052169$, which has a digital sum of 27.