MU ALPHA THETA

NATIONAL CONVENTION
SAN DIEGO 2013

Linear Algebra
Mu, Round 2
Test #411

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select “E) NOTA” and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select “E) NOTA” in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct. Furthermore, assume that $\det(A)$, and $\text{rref}(A)$ denote the determinant and row-reduced echelon form, respectively, of the matrix A.

Assume all answer choices have correct units.

Good luck!

1. Compute $\det\begin{pmatrix} -2 & 2 & 3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix}$.

(A) 3 (B) 4 (C) 5 (D) 6 (E) NOTA

2. Compute the sum of the entries of $\sum_{n=0}^{\infty}\begin{pmatrix} \frac{1}{2^n} & \frac{1}{3^n} \\ \frac{1}{4^n} & \frac{1}{5^n} \end{pmatrix}$.

(A) 1 (B) $\frac{25}{12}$ (C) $\frac{50}{12}$ (D) $\frac{73}{12}$ (E) NOTA

3. Compute $\text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ for $a, b \in \mathbb{R}$.

(A) $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ (B) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ (C) $\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$ (D) $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ (E) NOTA

4. Compute $t \in \mathbb{R}$ such that $\det\left[\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & t^2 \end{pmatrix}\right]$ is minimized in the range $-2 < t < 2$.

(A) $-\frac{1}{3}-\frac{\sqrt{13}}{3}$ (B) $\frac{1}{3}-\frac{\sqrt{13}}{3}$ (C) $\frac{1}{3}+\frac{\sqrt{13}}{3}$ (D) $\frac{1}{3}\pm\frac{\sqrt{13}}{3}$ (E) NOTA

5. Compute $\int_{-\pi}^{\pi}\left(1+\cos x+\cos 2x+\cdots+\cos(2013x)\right)\left(1+\sin x+\sin 2x+\cdots+\sin(2013x)\right)dx$.

(A) 0 (B) π (C) 2π (D) 2013π (E) NOTA

6. Let the inner product of the functions given by $f(x) = x+1$ and $g(x) = \frac{1}{x^2+1}$ over the interval $[0,1]$ be A. Compute A^2.

(A) $\frac{1}{4}\ln 2 + \frac{\pi}{4}$ (B) $\frac{1}{2}\ln 2 + \frac{\pi}{4}$ (C) $\ln 2 + \frac{\pi}{4}$ (D) $\frac{1}{2}\ln 2 + \pi$ (E) NOTA

7. An $n \times n$ matrix A satisfies $A_{i1} = i$ for $1 \le i \le n$ and $A_{i(j+1)} = A_{ij} + 1$ for $j \ne n$. For example, the 3×3 matrix would be $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$. Compute $\operatorname{tr}(A)$ in terms of n.

(A) $2n-1$ (B) $2n$ (C) n^2 (D) n^2+n (E) NOTA

8. Determine the smallest positive integer n such that there exists an integral linear combination of the entries in the set $\{x^2-1, x^2+5x+6, x+3\}$ which equals $n(2x^2+7x+8)$.

(A) 8 (B) 9 (C) 10 (D) 11 (E) NOTA

9. Ethan is constructing two n-tuple vectors U and V. For each entry in the vectors, he flips a fair two-sided coin and records 1 if the coin is heads and 0 if the coin is tails. Assuming the vectors are completed, compute the probability that the two vectors are orthogonal.

(A) $\left(\frac{3}{4}\right)^n$ (B) $\left(\frac{3}{4}\right)^n - \left(\frac{1}{4}\right)^n$ (C) $\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n$ (D) $\frac{1}{2}$ (E) NOTA

10. The matrix $A = \begin{pmatrix} -1 & 6 & 6 \\ -1 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ can be diagonalized by the matrix $P = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. Compute the trace of the diagonal matrix D which is determined by A and P.

(A) 5 (B) 6 (C) 7 (D) 8 (E) NOTA

11. Consider the set of vectors $\mathrm{v}=(v_1,v_2,v_3,v_4,v_5)$ in $\mathbb{R}^5$ such that $v_1=v_2$ and $v_3+v_4+v_5=0$. This set of vectors is a subspace of $\mathbb{R}^5$. Which of the following is a basis for this subspace?

(A) $\left\{\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\\-1\end{pmatrix}\right\}$

(B) $\left\{\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\-1\end{pmatrix}\right\}$

(C) $\left\{\begin{pmatrix}1\\1\\1\\-1\\-1\end{pmatrix}\right\}$

(D) $\left\{\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\-1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\-1\\0\\0\end{pmatrix}\right\}$

(E) NOTA

12. Let $A=\begin{bmatrix}1&2&0&1\\0&4&3&6\\2&0&1&0\\-1&2&1&3\\1&-2&1&-1\end{bmatrix}$. Compute $\operatorname{rref}(A)$.

(A) $\begin{pmatrix}1&0&0&-0.5\\0&1&0&0.75\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$

(B) $\begin{pmatrix}1&0&0&0.5\\0&1&0&-0.75\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$

(C) $\begin{pmatrix}1&0&0&0.5\\0&1&0&-0.75\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

(D) $\begin{pmatrix}1&0&0&0.5\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

(E) NOTA

13. Determine the rank of the matrix A given in Problem 12.

(A) 2 (B) 3 (C) 4 (D) 5 (E) NOTA

14. Consider an $n \times n$ matrix A. The sum of the eigenvalues of A is 23. Determine the sum of the eigenvalues of the matrix $C = B^{-1}AB$, where B is some nonsingular $n \times n$ matrix.

(A) 0 (B) $\frac{23}{2}$ (C) 23 (D) 46 (E) NOTA

15. The span

$$S = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -2 \\ 5 \end{pmatrix} \right\}$$

forms a subspace for $\mathbb{R}^4$. Let B be a basis for this subspace. Determine the dimension of B.

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

16. Call the vector space of 2×2 matrices $M_{2\times 2}$. Consider two vectors in this space $v_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. What choice of another vector in this space, v_3, would make the set $\{v_1, v_2, v_3\}$ linearly dependent?

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (E) NOTA

17. Which of the following is a basis for the null space of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$?

(A) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ (B) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ (C) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ (D) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ (E) NOTA

18. Determine the eigenvalues of the matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.

(A) $1 \pm 2i$ (B) $2 \pm 3i$ (C) $3 \pm 4i$ (D) $4 \pm 5i$ (E) NOTA

19. The functions $f_1(x)=x$ and $f_2(x)=x^2$ are orthogonal over $[-2,2]$. There exists constants c_1 and c_2 such that $f_3=x+c_1x^2+c_2x^3$ is orthogonal to both f_1 and f_2 over the same range. Compute the value of $12c_2$.

(A) -7 (B) -5 (C) 5 (D) 7 (E) NOTA

20. The *permanent* of a $n\times n$ matrix A, or $\operatorname{perm}(A)$, is a polynomial of the entries of a matrix, similar to the determinant. In general, it usually is much more difficult to compute that the determinant. We can, however, use an algorithm called the *Ryser formula* to compute the permanent. First, let A_k denote the matrix when we delete k columns from A. Then, let $P(A_k)$ denote the product of row-sums of A_k, and let S_k denote the sum of the values of $P(A_k)$ over all possible A_k. Then the permanent is

$$\operatorname{perm}(A)=\sum_{k=0}^{n-1}(-1)^k S_k$$

Consider a general matrix $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Deduce $\operatorname{perm}(B)$ in terms of $a,b,c,$ and d.

(A) $ab+cd$ (B) $ad+bc$ (C) $ac+bd$

(D) $(a+b)(c+d)$ (E) NOTA

<u>For Problems 21, 22, and 23, use the following information:</u>

Consider a vector space V consisting of all functions of the form $f(x)=e^{2x}(\alpha\cos x+\beta\sin x)$, where $\alpha,\beta\in\mathbb{R}$. Furthermore consider the linear transformation $L:V\to V$ such that $L(f)=f'+f$.

21. Let $g(x)=e^{2x}(2\cos x+3\sin x)$. Compute $L(g)$.

(A) $e^{2x}(2\cos x+3\sin x)$ (B) $e^{2x}(5\cos x+4\sin x)$

(C) $e^{2x}(7\cos x+4\sin x)$ (D) $e^{2x}(9\cos x+7\sin x)$ (E) NOTA

22. The vector space V has a basis of $\{e^{2x}\cos x, e^{2x}\sin x\}$. Determine the matrix representation of L with respect to this basis.

(A) $\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ (B) $\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$ (C) $\begin{pmatrix} -1 & 3 \\ 1 & 3 \end{pmatrix}$ (D) $\begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}$ (E) NOTA

23. Use the matrix you found in Problem 22 to find a solution to the differential equation $y' + y = e^{2x}\cos x$ in the form $y = a\cos x + b\sin x$ for reals a, b. Compute $\frac{a}{b}$.

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

24. Consider the vector space of all polynomials with degree less than or equal to n. A basis for this vector space is $B = \{1, x, x^2, \cdots, x^n\}$. Consider the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and its derivative $p'(x)$. We can represent this polynomial as a vector $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$ with respect to B. Deduce a matrix M such that $Mp = p'$.

(A) $M_{i,j} = \begin{cases} i, & \text{if } j = i+1 \\ 0, & \text{if } j \neq i+1 \end{cases}$ (B) $M_{i,j} = \begin{cases} j, & \text{if } j = i+1 \\ 0, & \text{if } j \neq i+1 \end{cases}$

(C) $M_{i,j} = \begin{cases} i, & \text{if } j = i+1 \\ j, & \text{if } j \neq i+1 \end{cases}$ (D) $M_{i,j} = \begin{cases} j, & \text{if } j = i+1 \\ i, & \text{if } j \neq i+1 \end{cases}$ (E) NOTA

25. Consider the basis $B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$ for $\mathbb{R}^2$. Determine the components of the vector $v = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ in terms of the basis B.

(A) $(19, -8)$ (B) $(20, -8)$ (C) $(21, -8)$ (D) $(22, -8)$ (E) NOTA

26. Find a set of vectors which spans the image mapped by the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2 = \begin{pmatrix} 3 & 6 & -3 \\ 1 & 2 & 1 \end{pmatrix}$.

(A) $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ (B) $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$

(C) $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$ (D) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} \right\}$ (E) NOTA

For Problems 27 and 28, use the following information:

Cauchy-Schwarz Inequality. For two vectors v_1 and v_2 with inner product $\langle v_1, v_2 \rangle$ and norms $\|v_1\|$ and $\|v_2\|$, we have

$$\left|\langle v_1, v_2 \rangle\right| \le \|v_1\| \cdot \|v_2\|,$$

where we have equality when v_1 and v_2 are collinear.

27. A vector $v = (x, y, z, 2)$ in $\mathbb{R}^4$ satisfies

$$3\left(x^2 + y^2 + z^2 + 4\right) = 2\left(xy + yz + xz\right) + 4\left(x + y + z\right).$$

Compute the value of xyz.

(A) 8 (B) 12 (C) 18 (D) 24 (E) NOTA

28. There exist numbers $a_1, a_2, \ldots, a_n$ such that $a_1^2 + a_2^2 + \cdots + a_n^2 = 2.$ Compute the maximum possible value of $a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n + a_n a_1.$

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$ (E) NOTA

For Problems 29 and 30, use the following information:

It is well known from the Taylor Series that for complex x, e^x can be written as $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$ However, x is not restricted to numbers. In fact, let A be a $n \times n$ matrix and I be the $n \times n$ identity matrix. Then we can write

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}.$$

29. Calculate e^A where $A = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ (*Hint: Multiply out a few* A^n *).*

(A) $\begin{pmatrix} 1 & 3 & 13 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 6 & 13 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & 6 & 13 \\ 6 & 1 & 3 \\ 13 & 3 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$ (E) NOTA

30. Define the *n diagonal matrix* as an $n \times n$ matrix A_n which satisfies $a_{ij} = -i$ for $i = j$ and $a_{ij} = 0$ elsewhere. Let $B_n = e^{A_n}$. Compute $\lim_{n\to\infty}\left[\operatorname{tr}(B_n)\right]$, where $\operatorname{tr}(M)$ denotes the trace of a matrix M.

(A) $\frac{1}{e(e-1)}$ (B) $\frac{1}{e}$ (C) $\frac{1}{e-1}$ (D) $\frac{e}{e-1}$ (E) NOTA