Logic & Set Theory
Open, Round 3
Test #601

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select "E) NOTA" and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select "E) NOTA" in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

Note: This test is based on the Zermelo-Fraenkel Axioms (ZF) of Set Theory.

1. If $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 5\}$, find $A \cup B$.

   (A) $\emptyset$ (B) $\{-2, -1, 0, 1, 2, 5\}$
   (C) 6 (D) $\{0, 1\}$ (E) NOTA

2. If $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 5\}$, find $A \cap B$.

   (A) $\{0, 1\}$ (B) $\emptyset$
   (C) $\{-2, -1, 0, 1, 2, 5\}$ (D) 2 (E) NOTA

3. For $X = \{0, 1, 2, \ldots, 2012\} = \{x \mid x \text{ is a nonnegative integer less than } 2013\}$, let $A \subset X$ be a set whose elements are all odd. Find the product of the elements of the complement of A with respect to X. In other words, the product of all the elements in X *not* in A.

   (A) 1 (B) 2^{2013} (C) 0 (D) $\frac{2013!}{2}$ (E) NOTA

4. Given sets A and B, the *set difference* $A - B$ is defined by $A - B = \{x \mid x \in A \wedge x \notin B\}$. If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and B is the set of prime numbers, find the sum of the elements of $A - B$.

   (A) 47 (B) 48 (C) 49 (D) 50 (E) NOTA

5. For set S, the set consisting of all subsets of S is called the *PowerSet* of S. How many elements are in the PowerSet of {0, 1, 2, 3, 4, 5}?

   (A) 30 (B) 62 (C) 32 (D) 64 (E) NOTA

6. For a statement S, let $\neg S$ be its negation. What is the contrapositive of $\neg P \to Q$?

   (A) $P \to \neg Q$ (B) $\neg Q \to P$ (C) $Q \to \neg P$ (D) $P \to Q$ (E) NOTA

7. Which of the following is the inverse of the statement below?

If it rains, I will study math.

(A) If it does not rain, I will not study math.

(B) If it rains, I will not study math.

(C) If I will study math, then it will not rain.

(D) If I don't study math, it will not rain.

(E) NOTA

8. Let x, y, and z be sets. Which of the following is the most accurate English translation of the statement below?

$$\forall x \forall y \exists z((z \subset x) \wedge (z \subset y))$$

(A) For all x and all y, there exists z such that z is a subset of the union of x and y.

(B) There exists x and y such that for all z, z is a subset of at least one of x or y.

(C) For all x and all y, there exists z such that z is a subset of x and y.

(D) There exists x and y such that there is a z that is an element of $x \cap y$.

(E) NOTA

9. Which of the following is the most accurate translation of the statement below using symbols of first-order logic? Assume that x and y are real numbers.

"For all positive x there exists a y such that $x = y^2$"

(A) $\exists x \forall y \rightarrow (x \ngtr y^2) \vee (x \nless y^2)$ (B) $(\exists x > 0) \rightarrow \forall y \rightarrow (x = y^2)$

(C) $\forall x \exists y \rightarrow (x^2 > 0) \wedge (x = y^2)$ (D) $\forall x(x > 0 \rightarrow \exists y(x = y^2))$ (E) NOTA

10. Which of the following sets can be used to define the *ordered pair* (a, b)?

(A) $\{a, b\}$ (B) $\{\{a\}, \{b\}\}$ (C) $\{a, \{a, b\}\}$ (D) $\{\{a, b\}\}$ (E) NOTA

11. How many of the following is/are true for all sets A, B, and C?

- $A - B = B - A$ if and only if $A = B$.
- If $A \cup B = A \cup C$, then $B = C$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup B \subset B$

(A) 4 (B) 3 (C) 2 (D) 1 (E) NOTA

12. For integer x, let $\phi(x)$ represent the statement "x is an even number." Let Z be the set of the 100 smallest positive integers. If $Y = \{y \in Z \mid \phi(y)\}$, find the sum of the elements of Y. Note that by the Axiom of Restricted Comprehension, the set Y exists.

(A) 2550 (B) 5000 (C) 2500 (D) 5050 (E) NOTA

13. Which of the following is the most accurate statement of what is most commonly known as *Russel's Paradox*?

(A) For set A, let $M = \{A \mid A \notin A\}$. The set M is simultaneously an element and not an element of M.

(B) A sphere can be decomposed and then reconstructed into two spheres each having the same volume as the original sphere.

(C) Every countable axiomatization of set theory using first-order logic, if consistent, contains a countable model.

(D) Motion is impossible. To move a certain distance, one must first move half the distance. But to move half the distance, one must move a quarter of the distance, etc.

(E) NOTA

14. Which of the following is/are equivalent to the Axiom of Choice?

I. Zorn's Lemma
II. Every set has a Well-Ordering.
III. Every vector space has a basis.
IV. If S is an infinite set, S has the same cardinality as $S \times S$.

(A) None (B) I, II, III, IV (C) I & II only (D) I, II, III only (E) NOTA

15. For sets A and B, their *Cartesian Product* $A \times B$ is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

If A is a set with 4 elements and B is a set with 3 elements, how many elements does $A \times B$ have?

(A) 7 (B) 12 (C) 64 (D) 81 (E) NOTA

16. Find the number of functions $f: A \to B$, where the domain of f is $A = \{0, 1, 2, 3\}$ and the range of f is $B = \{-1, 0, 1\}$.

(A) 81 (B) 64 (C) 7 (D) 12 (E) NOTA

17. Determine the cardinality of a set whose only elements are the terms of the arithmetic sequence given by $-65, -58, \dots, 628$.

(A) 100 (B) 1024 (C) 520 (D) 693 (E) NOTA

18. Let $X = \{\emptyset\}$ and Y be the PowerSet of X. How many of the following are elements of the PowerSet of Y?

I. $\emptyset$
II. X
III. $\{X\}$
IV. $\{\emptyset, X\}$

(A) 3 (B) 4 (C) 2 (D) 1 (E) NOTA

19. Denote the cardinality of set S by $|S|$. For sets A, B, and C, if $|A| = 50$, $|B| = 70$, $|C| = 60$, and $|A \cup B \cup C| = 120$, find the largest possible cardinality of $A \cap B \cap C$.

(A) 30 (B) 24 (C) 20 (D) 12 (E) NOTA

20. What is the sum of all integers x such that the interval

$$[x^2 + 18, 11x - 6) = \{r \in \mathbb{R} \mid (r \geq x^2 + 18) \wedge (r < 11x - 6)\}$$

has the same cardinality as the set of real numbers $\mathbb{R}$?

(A) 44 (B) 33 (C) 11 (D) 22 (E) NOTA

21. Define a sequence of sets as $S_0 = \emptyset$ and for integers $n \geq 0$, $S_{n+1} = S_n \cup \{S_n\}$. Find S_3.

(A) $\{\{\{\emptyset\}\}\}$ (B) $\{\{\emptyset, \{\emptyset\}\}\}$

(C) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ (D) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ (E) NOTA

22. Consider the following order relation $\leq_d$ on the set of positive integers defined by the following: $a \leq_d b$ if and only if b is divisible by a. Which of the following is/are true?

I. The relation $\leq_d$ is a linear order.
II. The relation $\leq_d$ is a partial order.
III. The relation $\leq_d$ is an equivalence relation.
IV. The relation $\leq_d$ induces a well-ordering on the set of integers.

(A) II only (B) I and II (C) I and IV (D) I, II, III, IV (E) NOTA

23. Which of the following is the most accurate statement of the Axiom of Extensionality?

(A) A nonempty set X contains an element Y that is disjoint from X.

(B) Given any set A with real number entries, we can construct a superset B, also having elements that are real numbers, such that any sequence in B with an upper bound contains a least upper bound.

(C) There exists a set with infinitely many elements.

(D) Two sets are equal if and only if they have the same elements.

(E) NOTA

24. A *topology* T on a set X is a collection of subsets of X such that $\emptyset \in T$, $X \in T$, the union of any number of elements of T is an element of T, and an intersection of a finite number of elements of T is an element of T. For how many of the following is T a topology on X?

I. For any set X, $T = \{\emptyset, X\}$.
II. For any set X, T is the PowerSet of X.
III. If $X = \{a, b\}$, $T = \{\emptyset, \{a\}, \{b\}\}$.
IV. If $X = \{a, b\}$, $T = \{\emptyset, X, \{a\}\}$.
V. If $X = \{a, b, c\}$, $T = \{X, \{a\}, \{b\}, \{c\}, \{a, b\}\}$.
VI. If $X = \{a, b, c\}$, $T = \{\emptyset, X, \{a, b\}\}$.
VII. If $X = \{a, b, c\}$, $T = \{\emptyset, X, \{b, c\}, \{a\}\}$.

(A) 7 (B) 5 (C) 3 (D) 2 (E) NOTA

25. Suppose a statement P is *independent* of the axioms of Zermelo Fraenkel Set Theory (hereby abbreviated as "ZF"). Which of the following is necessarily true?

I. If a proof of a theorem invokes P, a different proof of the theorem must also invoke P.
II. Statement P cannot be proved nor disproved using the axioms of ZF.
III. Any theorem provable with the axioms of ZF is provable using P alone.
IV. The axioms of ZF combined with P imply the existence of Large Cardinals.

(A) I only (B) II only (C) III only (D) IV only (E) NOTA

26. Let $S = \{1, 2, 3, 4, 5\}$. Find the number of ordered triples of sets (S_1, S_2, S_3) such that $S_1 \cup S_2 \cup S_3 = S$ and $S_1 \cap S_2 \cap S_3 = \emptyset$.

(A) 7776 (B) 1024 (C) 15625 (D) 10000 (E) NOTA

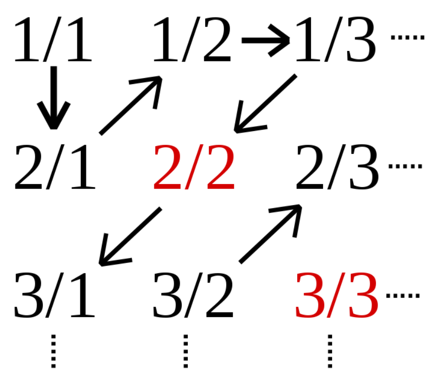
27. Which of the following sets S makes the following true: $|S \times S| + 2|S| + |\mathbb{Q}| = |\mathbb{R}|$. Assume the Axiom of Choice.

(A) $\mathbb{Q}$ (B) $\mathbb{Q} \times \mathbb{Q}$ (C) $\mathbb{R}^3$ (D) $\mathbb{N} \times \mathbb{Q}^2$ (E) NOTA

28. Find the number of subsets A of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ such that A has exactly five elements and the sum of all the elements in A is divisible by 5.

(A) 58 (B) 56 (C) 54 (D) 52 (E) NOTA

29. A classic proof that the set of rational numbers $\mathbb{Q}$ has the same cardinality as the set of positive integers $\mathbb{N}$ involves arranging the positive elements of $\mathbb{Q}$ in an infinitely large grid such that the number m/n (not necessarily reduced) corresponds to the m^{th} row, n^{th} column entry in the grid. Once the positive rational numbers are listed out in this manner, we make a one-to-one correspondence with $\mathbb{N}$ by "zig zagging" along the grid and listing out the numbers encountered, omitting numbers that have already been listed. This process is illustrated in the diagram below:



This produces the sequence $1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ (notice that $\frac{2}{2} = 1$ is not listed, as that number is already in the sequence). Find the 25^{th} term of this sequence.

(A) $\frac{7}{3}$ (B) $\frac{4}{5}$ (C) $\frac{5}{4}$ (D) $\frac{3}{7}$ (E) NOTA

30. Let α be a transfinite ordinal. In other words, an ordinal with an order type greater than the order type of the set of positive integers ω. Which of the following is equivalent to the ordinal $(1 + \alpha) \cdot 2 + \alpha + 2 \cdot \alpha$?

(A) $\alpha \cdot 4$ (B) $\alpha + \alpha$ (C) $5 \cdot \alpha + 2$ (D) $2 + \alpha \cdot 2 + 3 \cdot \alpha$

(E) NOTA