Theta Complex Numbers Solutions

1. Calculate $(-5+2i)-(12-4i)$.

$$\begin{aligned}(-5+2i)-(12-4i) &= -5+2i-12+4i\\ &= -5-12+2i+4i\\ &= -17+6i.\end{aligned}$$

B.

2. Solve for x: $|24-7i| = x|20+21i|$.

$|24-7i| = \sqrt{(24)^2+(-7)^2} = \sqrt{576+49} = \sqrt{625} = 25$
$|20+21i| = \sqrt{(20)^2+(21)^2} = \sqrt{400+441} = \sqrt{841} = 29$

$|24-7i| = x|20+21i|$
$x = \dfrac{|24-7i|}{|20+21i|} = \dfrac{25}{29}$
A.

3. Simplify $-\frac{5\sqrt{18}}{\sqrt{-45}}$

$$\begin{aligned}-\frac{5\sqrt{18}}{\sqrt{-45}} &= -\frac{15\sqrt{2}}{3\sqrt{-1}\sqrt{5}}\\ &= -\frac{15\sqrt{10}}{15i}\\ &= i\sqrt{10}\end{aligned}$$

C.

4. Solve for $z = a+ib$ where $a = b-2$ is real and $z^2(z-2i) = 2z$.

$z^2(z-2i) = 2z.$
$z^3 - 2iz^2 - 2z = 0$
$z = 0$ or $z = \frac{2i \pm \sqrt{(-2i)^2-4(1)(-2)}}{2} = i \pm \frac{\sqrt{-4+8}}{2} = i \pm 1$
Since $a = b-2$, that leaves $z = -1+i$ as the answer.
B.

5. Calculate $(i\sqrt{3}+1)^3$.

$\left(i\sqrt{3}+1\right)^3 = \left(2e^{i\pi/3}\right)^3 = 8e^{i\pi} = -8$
D.

6. Find the distance between the points $4-10i$ and $3i+7$ in the complex plane.

Distance: $\sqrt{(4-7)^2+(-10-3)\text{^}2} = \sqrt{9+169} = \sqrt{178}$
C.

~~7. Simplify $\frac{5i-3}{1+i}$.~~

$$\frac{5i-3}{1+i} = \frac{5i-3}{1+i}\left(\frac{1-i}{1-i}\right) = \frac{-3+5+5i+3i}{2} = \frac{2+8i}{2} = 1+4i$$

~~C.~~

~~8. Compute the product of the values of z, where z satisfies the infinite series $1 = z^2(\frac{i}{2} + z^2(\frac{i}{2} + z^2(\frac{i}{2} + \cdots$~~

$$1 = z^2(\frac{i}{2} + 1)$$
$$2 = z^2(i+2)$$
$$z^2 = \frac{2}{i+2}\frac{2-i}{2-i} = \frac{4-2i}{5}$$

~~D.~~

9. If $z = 9 - 12i$ calculate $\frac{z}{\sqrt{\bar{z}z}}$ where $\bar{z}$ denotes the conjugate of z.

$$\frac{z}{\sqrt{\bar{z}z}} = \frac{9-12i}{\sqrt{(9+12i)(9-12i)}} = \frac{9-12i}{\sqrt{81+144}} = \frac{9-12i}{15} = \frac{3-4i}{5}$$

B.

10. If $Im(z) = 6$, $|z| = 15$, and z is in the second quadrant on the complex plane, what is $Re(z)$?

$$|z| = \sqrt{Im(z)^2 + Re(z)^2} = \sqrt{6^2 + Re(z)^2} = 15$$
$$36 + Re(z)^2 = 225$$
$$Re(z)^2 = 189$$
$$|Re(z)| = 3\sqrt{21}$$

Since z is in the second quadrant,

$$Re(z) = -3\sqrt{21}$$

A.

11. Which of the following are true?

I) A complex number with a non-zero real and imaginary component's square can be real.
II) The product of two complex numbers in the first quadrant must be in the first quadrant.
III) The quotient of two complex numbers in the first quadrant must be in the first or fourth quadrant.

The square root of any real number is either completely real (if it is positive) or completely imaginary (if it is negative). The contra-positive of I is false, so I is false.

Take $z = 1 + 2i$ in the first quadrant.
$z^2 = (1+2i)^2 = 1 - 4 + 2i + 2i = -3 + 4i$ which is in the second quadrant.
By contradiction, II is false.

Take $x = a\ cisb$ and $y = c\ cisd$, x and y in the first quadrant so $0 < b, d < {}^{\pi}/_{2}$

Then $\frac{x}{y} = \frac{a}{c} cis(b-d)$, so $-{}^{\pi}/_{2} < b - d < {}^{\pi}/_{2}$.
$\frac{x}{y}$ is in the first of fourth quadrant, so III is true.
C.

12. Calculate the reciprocal of $1 + 4i$.

$$\frac{1}{1+4i} = \frac{1}{1+4i}\left(\frac{1-4i}{1-4i}\right) = \frac{1-4i}{17}$$

A.

13. What is the imaginary component of $(2+2i)^4$?

$(2+2i)^4 = \left(2\sqrt{2}\ cis\frac{\pi}{4}\right)^4 = 64\ cis\pi$, which has no imaginary component.
C.

~~14. How many 6th roots of unity are there which have nonzero imaginary components?~~

~~The 6th roots of unity are $cis0, cis\frac{\pi}{3}, cis\frac{2\pi}{3}, cis\pi, cis\frac{4\pi}{3}, cis\frac{5\pi}{3}$.~~
~~All have imaginary components except $cis0$ and $cis\pi$ so 4 roots of unity have nonzero imaginary components.~~

15. Compute the sum of the magnitudes of the solutions of $z - 5 = \sqrt{-10z + 4z\sqrt{-5}}$.

$z^2 - 10z + 25 = -10z + 4z\sqrt{-5}$
$z^4 + 50z^2 + 25^2 = 16z^2(-5)$
$z^4 + 130z^2 + 25^2 = 0 = (z^2+5)(z^2+125)$
$z^2 = -5, -125.\ z = \pm\sqrt{5}, \pm 5\sqrt{5}$
$\sqrt{5} + \sqrt{5} + 5\sqrt{5} + 5\sqrt{5} = 12\sqrt{5}$
D.

16. Calculate $\left(\frac{7-3i}{1+i}\right)\left(\frac{6-5i}{4-10i}\right)$.

$$\left(\frac{7-3i}{1+i}\right)\left(\frac{6-5i}{4-10i}\right) = \frac{42-15-35i-18i}{4+10-10i+4i} = \frac{27-53i}{14-6i}\left(\frac{7+3i}{7+3i}\right) = \frac{189+159+81i-371i}{116}$$
$$= \frac{348-290i}{116} = \frac{174-145i}{58} = \frac{6-5i}{2}$$

C.

17. Find the sum of the converging geometric series $1 + \frac{i}{3} - \frac{1}{9} - \frac{i}{27} + \cdots$

$$1 + \frac{i}{3} - \frac{1}{9} - \frac{i}{27} + \cdots = \frac{1}{1-\frac{i}{3}} = \frac{3}{3-i}\left(\frac{3+i}{3+i}\right) = \frac{9+3i}{10}$$

B.

~~18. What is the product of all the unique 5th roots of unity?~~

The 5^{th} roots of unity are $cis0, cis\frac{2\pi}{5}, cis\frac{4\pi}{5}, cis\pi, cis\frac{6\pi}{5}, cis\frac{8\pi}{5}$.

$$(cis0)\left(cis\frac{2\pi}{5}\right)\left(cis\frac{4\pi}{5}\right)\left(cis\frac{6\pi}{5}\right)\left(cis\frac{8\pi}{5}\right) = cis\frac{20\pi}{5} = cis0 = 1$$

D.

19. Let $z^4 = 4i, z = a + ib$ where a and b are real. What is the value of the product ab?

$z = a + ib$, a and b real.

$z^4 = 4i = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$

$a^4 - 6a^2b^2 + b^4 = 0, 4a^3b - 4ab^3 = 4 = 4ab(a^2 - b^2)$

$$a^2 - b^2 = \frac{1}{ab}$$

$$a^4 - 6a^2b^2 + b^4 = (a^2 - b^2)^2 - 4a^2b^2 = \left(\frac{1}{ab}\right)^2 - 4a^2b^2 = 0$$

$$\left(\frac{1}{ab}\right)^2 = 4a^2b^2$$

$$a^4b^4 = \frac{1}{4}$$

$ab = \pm\frac{\sqrt{2}}{2}$ since a and b are real

A.

20. Calculate $(\sqrt{6} + i\sqrt{2})^8$

$$(\sqrt{6} + i\sqrt{2})^8 = \left(2\sqrt{2}\left(cis\frac{\pi}{6}\right)\right)^8 = 4096cis\frac{4\pi}{3} = -2048 - 2048i\sqrt{3}$$

A.

21. Solve for real x and y: $3(3 - 2i) - (1 - 8i) = x(1 + i) - y(i - 1)$. $(x, y) =$

$3(3 - 2i) - (1 - 8i) = x(1 + i) - y(i - 1)$

$9 - 6i - 1 + 8i = x + ix + y - iy$

$8 + 2i = x + y + ix - iy$

$8 = x + y$

$2 = x - y$

$x = 5$, $y = 3$

D.

22. What is the product of all the unique n^{th} roots of unity?

The n^{th} roots of unity satisfy $z^n = 1$ or $z^n - 1 = 0$.

The product of the roots then equals $(-1)^n\frac{-1}{1} = (-1)^{n+1}$.

D.

23. Calculate the sum of the roots of $(z + z^2)^2 = 15(3z^2 + 2)(z^2 + 2z)$.

$(z + z^2)^2 = 15(3z^2 + 2)(z^2 + 2z)$

$z^4 + 2z^3 + z^2 = 15(3z^4 + 6z^3 + 2z^2 + 4z)$
$44z^4 + 88z^3 + z^2 + 4z = 0$
The sum of the roots is $-\frac{88}{44} = -2$.
B.

24. Let x, y, and z be the three prime factors of 2013 where $0 < x < y < z$. Calculate $i^{2x} + i^y + i^z$.

$2013 = 3 \times 11 \times 61$
$x = 3, y = 11, z = 61$
$i^{2x} + i^y + i^z = i^6 + i^{11} + i^{61} = -1 - i + i = -1$
D.

~~25. Evaluate $(1+i)^{212}$~~

~~$(1+i)^{212} = \left(\sqrt{2}cis\frac{\pi}{4}\right)^{212} = 2^{106}cis(53\pi) = -2^{106}$~~
~~D.~~

26. For what real values of c will $y = 3z^2 + 5z + c^2$ have no real roots?

$y = 3z^2 + 5z + c^2$ has no real roots when the discriminant is negative.
$0 > (5)^2 - 4(3)(c^2) = 25 - 12c^2$
$c^2 > \frac{25}{12}$
$|c| > \sqrt{\frac{25}{12}} = \frac{5\sqrt{3}}{6}$
B.

27. If $\frac{2-5\sqrt{3}+5i+2i\sqrt{3}}{2+5i} = z^2$ where a and b are real, which is a possible value of z?

$$\frac{2-5\sqrt{3}+5i+2i\sqrt{3}}{2+5i} = \frac{2+5i+i\sqrt{3}(2+5i)}{2+5i} = 1 + i\sqrt{3} = z^2$$
A: $(\sqrt{3}+i)^2 = 2 + 2i\sqrt{3}$. Incorrect
B: $(1+i\sqrt{3})^2 = -2 + 2i\sqrt{3}$. Incorrect
C: $(\frac{\sqrt{6}}{2} + \frac{i\sqrt{2}}{2})^2 = 1 + i\sqrt{3}$. Correct
D: $(\frac{\sqrt{2}}{2} + \frac{i\sqrt{6}}{2})^2 = -1 + i\sqrt{3}$. Incorrect
C.

~~28. How many 15th roots of unity are in the 2nd quadrant?~~

~~The 15th roots of unity are $cis0, cis\frac{2\pi}{15}, cis\frac{4\pi}{15}, \dots cis\frac{28\pi}{n}$.~~
~~$cis\frac{8\pi}{15}, cis\frac{10\pi}{15}, cis\frac{12\pi}{15}$, and $cis\frac{14\pi}{15}$ are in the 2nd quadrant. 4.~~
~~C.~~

~~29. Calculate $\left|\frac{5-i}{4+7i}\right|$.~~

~~$$\left|\frac{5-i}{4+7i}\right| = \frac{\sqrt{(5)^2+(-1)^2}}{\sqrt{(4)^2+(7)^2}} = \frac{\sqrt{26}}{\sqrt{65}} = \frac{\sqrt{10}}{5}$$~~

~~A.~~

30. Calculate $\left(1+i\sqrt{3}\right)^4(2-2i\sqrt{3})^3$

$$\left(1+i\sqrt{3}\right)^4(2-2i\sqrt{3})^3 = \left(\left(1+i\sqrt{3}\right)\left(2-2i\sqrt{3}\right)\right)^3\left(1+\sqrt{3}\right) = (8)^3\left(1+i\sqrt{3}\right) = 512+512i\sqrt{3}$$

Mu Alpha Theta National Convention: San Diego, 2013
Theta Complex Numbers Test - Updated Solutions

7 **(C)**. Take each addend and multiply top and bottom by the conjugate:

$$\frac{5+5i}{3-4i}+\frac{20}{4+3i}=\frac{(5+5i)(3+4i)}{16+9}+\frac{20(4-3i)}{16+9}=\frac{75-25i}{25}=3-i.$$

8 **(E)**. The points are $(3,-7)$, $(10,0)$, $(-4,1)$. By the determinant formula for the area of a triangle,

$$A=\frac{1}{2}\begin{vmatrix} 3 & -7 & 1 \\ 10 & 0 & 1 \\ -4 & 1 & 1 \end{vmatrix}=\frac{105}{2}.$$

So that $m+n=105+2=107$.

14 **(B)**. We have $|z|^2=m^2+9n^2$. We take a look at this expression in modulo 4. Perfect squares are congruent to either 0 or 1 in modulo 4. If m and n are both even, then clearly $|z|^2\equiv 0 \mod 4$. If m and n are both odd, then $|z|^2\equiv 10\equiv 2 \mod 4$. If m and n differ in parity, then $|z|^2\equiv 1 \mod 4$. Since $2011\equiv 3 \mod 4$, it cannot be a possible value for $|z|^2$.

18 **(D)**. Let $z=a+bi$, so that $z^2+2|z|^2=2$ becomes $a^2-b^2+2abi+2(a^2+b^2)=3a^2+b^2=2+0i$. Thus, $3a^2+b^2=2$ or $2ab=0$. If $a=0$, then $b=\pm\sqrt{2}$. If $b=0$, then $a=\pm\sqrt{2/3}$. Thus, the product of the absolute values of the solutions are $\sqrt{2}\sqrt{2}\sqrt{2/3}\sqrt{2/3}=2(2/3)=4/3$.

19 **(A)**. The roots come in conjugate pairs. Thus, $1-i$ and $-3i$ are also roots. This means that the polynomial's factorization is $P(x)=(x^2-2x+2)(x^2+9)$, making $P(2)=26$.

22 **(D)**. We know that $a^2+b^2=1$. Thus

$$N=\frac{z-1}{z+1}=\frac{a+bi-1}{a+bi+1}=\frac{a^2+b^2-1+2bi}{a^2+2a+1+b^2}=\frac{2bi}{a^2+2a+1+b^2},$$

so N is a purely imaginary number. The desired ratio is equal to 0.

25 **(D)**. We have $n=37^k\equiv(36+1)^k\equiv 1^k\equiv 1 \mod 4$. Thus, $i^n=i^1=i$.

28 **(C)**. If $z=x+yi$, then, by the Binomial Theorem, $z^3=(x+yi)^3=(x^2-y^2+2xyi)(x+yi)=(x^3-3xy^2)+(3x^2y-y^3)i=18+26i$, so that $x^3-3xy^2=18$ and $3x^2y-y^3=26$, yielding $18(3x^2y-y^3)=26(x^3-3xy^2)$. Set $y=tx$; note that since x and y are positive integers, t must be a nonzero rational number. Simplify and factor to obtain $(3t-1)(3t^2-12t-13)=0$. The only rational solution to this equation is $t=1/3$. This yields $x=1$ and $y=3$, making $x^5+y^5=1^5+3^5=1+243=244$.

29 **(A)**. The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = x^2 - 2x + 5 = 0.$$

which has roots of $1 \pm 2i$.