Answers:

1. C
2. B
3. D
4. A
5. A
6. E
7. B
8. C
9. B
10. A
11. C
12. D
13. E
14. C
15. B
16. D
17. A
18. C
19. A
20. D
21. A
22. A
23. B
24. C
25. A
26. C
27. B
28. D
29. D
30. A

TB1. 43
TB2. 19
TB3. 17

Solutions:

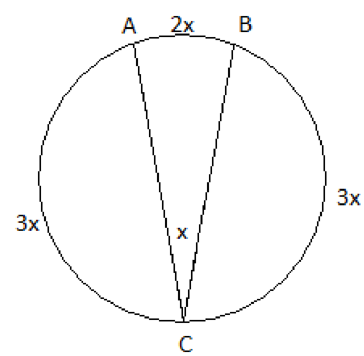
1. A square is a specific kind of rectangle, which is a specific kind of parallelogram, which is a specific kind of quadrilateral, which is a specific kind of polygon.

2. $f\left(f\left(f(-1)\right)\right)=f\left(f(5)\right)=f(-13)=41$

3. For $f(x)=\dfrac{20x^3+31x^2-31x-42}{15x^2+17x+2}$, since the degree on bottom is smaller than the degree on top, the function has no horizontal asymptotes, ruling out IV. Also, $\dfrac{20x^3+31x^2-31x-42}{15x^2+17x+2}$ $=\dfrac{(x+1)(4x+7)(5x-6)}{(x+1)(15x+2)}$, so the only vertical asymptote is $x=-\dfrac{2}{15}$ (II). Finally,

$\dfrac{20x^3+31x^2-31x-42}{15x^2+17x+2}=\dfrac{4}{3}x+\dfrac{5}{9}+\dfrac{-\frac{388}{9}x-\frac{388}{9}}{15x^2+17x+2}$, so $y=\dfrac{4}{3}x+\dfrac{5}{9}$ (V) is an oblique asymptote. Therefore, the answer is II and V only.

4. Using Cramer's Rule, $z=\dfrac{\begin{vmatrix}1&2&3\\4&0&6\\0&7&9\end{vmatrix}}{\begin{vmatrix}1&2&0\\4&0&5\\0&7&8\end{vmatrix}}=\dfrac{0+0+84-0-42-72}{0+0+0-0-35-64}=\dfrac{-30}{-99}=\dfrac{10}{33}$



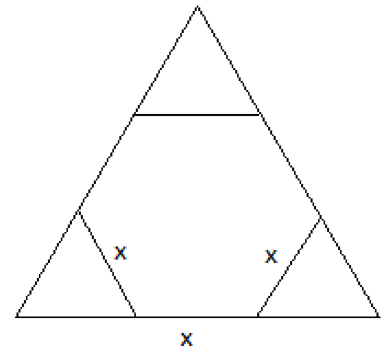
5. Since $\overline{AC}$ and $\overline{BC}$ are equal, minor arcs AC and BC are also equal. By the picture, $3x+2x+3x=8x=360^\circ \Rightarrow x=45^\circ$. Therefore, major arc ABC has measure $5x=225^\circ$.

6. $-6x=(y+8)(y-4)\Rightarrow x=-\dfrac{1}{6}(y+2)^2+6$, so the vertex is at the point $(6,-2)$.

7. Duplicate E's and L's, so there are $\dfrac{8!}{2!2!}=\dfrac{40{,}320}{4}=10{,}080$ distinct permutations.

8. The region is a torus where the radius of a cross-section is $\sqrt{7}$ and the radius from the center of the hole to the center of a cross-section is 4, so the volume is $2\pi^2 \cdot 4 \cdot \left(\sqrt{7}\right)^2 = 56\pi^2$.

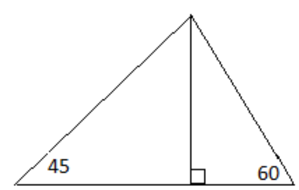
9. The hexagon encloses an area of $\frac{3x^2\sqrt{3}}{2}$. Since the equilateral triangle has side length $3x = 16$, the hexagon encloses an area of $\frac{3\sqrt{3}}{2}x^2 = \frac{3\sqrt{3}}{2}\left(\frac{16}{3}\right)^2 = \frac{128\sqrt{3}}{3}$.



10. The distance from P to Q is $\sqrt{(1+2)^2+(5-3)^2+(5+1)^2}$ $= \sqrt{9+4+36} = 7$. By the triangle inequality, the radius length, 12, is less than or equal to 7 + the distance to any point on the sphere, so the shortest distance is the remaining length of the radius, which is 5.

11. $0 = \begin{vmatrix} 2 & x & 1 \\ 1 & -x & 4 \\ x & 0 & -5 \end{vmatrix} = 10x + 4x^2 + 0 + x^2 - 0 + 5x = 5x^2 + 15x = 5x(x+3)$, so the nonzero solution is -3 (you could also notice that negating each of the first two rows, then adding them results in the third row for the second and third columns, so it should work for the first column also).

12. Drop the altitude from the 75° angle as in the picture, creating 45-45-90 and 30-60-90 right triangles. The hypotenuse of the 45-45-90 triangle is $4\sqrt{6}$, making the altitude of the large triangle $4\sqrt{3}$. This makes the base of the large triangle $4+4\sqrt{3}$, and thus the enclosed area is $\frac{1}{2}\left(4+4\sqrt{3}\right)\left(4\sqrt{3}\right) = 24 + 8\sqrt{3}$.



13. The lines could be skew or parallel but don't have to be either. The lines could intersect the third line in the exact same point, but again they do not have to do so. They won't be perpendicular, so none of the choices must describe the relationship between the two lines.

14. $\left|\overline{AE}\right| \cdot \left|\overline{BE}\right| = \left|\overline{CE}\right| \cdot \left|\overline{DE}\right| \Rightarrow \left|\overline{DE}\right| = \frac{6 \cdot 35}{15} = 14$

15. Each product is one of the terms in the expansion of $(1+2+6+10)(2+7+9+11)(1+3+5+14)$, which equals $19 \cdot 29 \cdot 23 = 12{,}673$.

16. The total points scored on the first three tests is $3\cdot 92=276$. The total points scored on all four tests is $4\cdot 94=376$. Therefore, the fourth test score is $376-276=100$.

17. Let A be the angle measure of the nine equivalent interior angles, and let B be the other interior angle. Then $1440=180(10-2)=9A+B$. Since the decagon is convex, we must have $1440=9A+B<9A+180\Rightarrow 9A>1260\Rightarrow A>140$. Since the angles must have integer degree values, the smallest A could be is 141° (if $A=140^\circ$, then $B=180^\circ$, which wouldn't be convex).

18. $5=45r^2\Rightarrow r=\pm\frac{1}{3}$. Since the first term is $\frac{45}{1/9}=405$, the sum of all such series is

$$\frac{405}{1-\frac{1}{3}}+\frac{405}{1+\frac{1}{3}}=\frac{1215}{2}+\frac{1215}{4}=\frac{3645}{4}.$$

19. This is the same number as that enclosed by $x^2+y^2\le 25$. There is one point each for $x=\pm 5$; 7 points each for $x=\pm 4$; 9 points each for $x=\pm 3$, $x=\pm 2$, and $x=\pm 1$, and 11 points for $x=0$. Thus there are $2+14+54+11=81$ points.

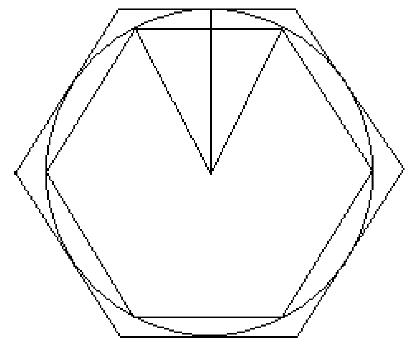
20. The cylinder has volume $\pi(8)^2(5)=320\pi$, and each ball has volume $\frac{4}{3}\pi(2)^3=\frac{32}{3}\pi$. Therefore, the number of balls that would fit into the cylinder is $\frac{320\pi}{32\pi/3}=30$.

21. The equation is the same as $\left(x^2\right)^2=(x+1)^2$, so the solutions to the equation are solutions to either $x^2=x+1$ or $x^2=-x-1$. The first equation, which is also $x^2-x-1=0$, has solutions $x=\frac{1\pm\sqrt{5}}{2}$. The second equation, which is also $x^2+x+1=0$, has solutions $x=\frac{-1\pm\sqrt{3}i}{2}$. The only positive solution is $\frac{1+\sqrt{5}}{2}$.

22. Let $S=\sum_{n=1}^{\infty}\left((2n+3)\left(\frac{2}{3}\right)^n\right)=5\left(\frac{2}{3}\right)+7\left(\frac{2}{3}\right)^2+9\left(\frac{2}{3}\right)^3+\ldots$. Multiply this equation by $\frac{2}{3}$ to get $\frac{2}{3}S=5\left(\frac{2}{3}\right)^2+7\left(\frac{2}{3}\right)^3+\ldots$. Now, subtract this second equation from the first equation to get

$\frac{1}{3}S=5\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)^2+2\left(\frac{2}{3}\right)^3+\ldots=\frac{10}{3}+\frac{8/9}{1-2/3}=\frac{10}{3}+\frac{8}{3}=\frac{18}{3}=6$. Therefore, $S=18$.

23. The radius of the circle is $\frac{1}{2}$, so this is also the side length of the inscribed hexagon and the apothem of the circumscribed hexagon. Therefore, the perimeters of the inscribed and circumscribed hexagons, respectively, are $6\left(\frac{1}{2}\right)=3$ and $6\left(2\cdot\frac{1}{2\sqrt{3}}\right)=2\sqrt{3}$. Since the circumference of the circle is $2\pi\left(\frac{1}{2}\right)=\pi$, and since the perimeter of the hexagons are less than and greater than the circumference of the circle, we must have $3<\pi<2\sqrt{3}$.



24. $A\to(B\vee C)$ is logically equivalent to $\sim A\vee(B\vee C)$, so knowing $\sim A$ (I) is True would be sufficient. Since $\sim B\to\ \sim C$ (II) is logically equivalent to $B\vee\sim C$, knowing this is True means at least one of B or $\sim C$ is true, but this is not sufficient since B and C could both be false and A could be true, making the statement False. Since $\sim B\to\ \sim A$ (III) is logically equivalent to $B\vee\sim A$, knowing this is True means at least one of B or $\sim A$ is True. If B is True, the original statement is True; if $\sim A$ is True, the original statement is True. Therefore, knowing III is True would be sufficient as well. Thus, I & III would be sufficient but II would not.

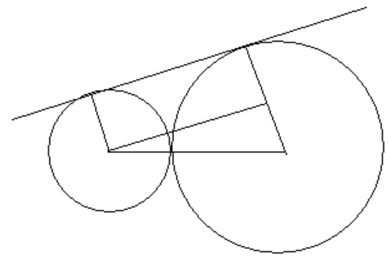
25. The largest triangle drawn on a hemisphere would be if all of the vertices are on the great circle of the hemisphere. All angles would thus be 180°, making the maximum sum total 540°.

26. The five points make a pentagon, so we are looking for the product of the two diagonals and two of the edges of the pentagon. Both diagonals are equal, and a right triangle can be formed with the diagonal (z) as the hypotenuse, half the side length of the pentagon (x) as one leg, and the distance from a vertex to the side opposite it (y) as the other leg. The angle between this second leg and the hypotenuse has measure of 18°. $y=1+\cos 36^\circ$ and $x=\sin 36^\circ$, so $z^2=x^2+y^2=\sin^2 36^\circ+1+2\cos 36^\circ+\cos^2 36^\circ=2\left(1+\cos 36^\circ\right)=4\cos^2 18^\circ$.
Multiply this by $(2x)^2=4\sin^2 36^\circ=16\sin^2 18^\circ\cos^2 18^\circ$, you get $64\sin^2 18^\circ\cos^4 18^\circ$, and since

$\sin 18^\circ = \frac{-1+\sqrt{5}}{4}$, $\sin^2 18^\circ = \frac{3-\sqrt{5}}{8}$ and $\cos^2 18^\circ = \frac{5+\sqrt{5}}{8} \Rightarrow \cos^4 18^\circ = \frac{5\left(3+\sqrt{5}\right)}{32}$,

$64\sin^2 18^\circ \cos^4 18^\circ = 320\left(\frac{3-\sqrt{5}}{8}\right)\left(\frac{3+\sqrt{5}}{32}\right) = 5$.

27. Let a and b be the lengths of the smaller and larger radii, respectively. Based on the picture, the right triangle whose hypotenuse is the sum of the radii of the circles satisfies $(b+a)^2 = b^2 + (b-a)^2 \Rightarrow b^2 = 4ab \Rightarrow b = 4a$. Therefore, the ratio of the larger radius to the smaller radius is $4:1$.



28. $\frac{0+1+2+3+...+2012}{2013} = \frac{\frac{2012\cdot 2013}{2}}{2013} = 1006$

29. $\sqrt{t+2} = \frac{t+3}{t-1} \Rightarrow t+2 = \frac{t^2+6t+9}{t^2-2t+1} \Rightarrow t^3-3t+2 = t^2+6t+9 \Rightarrow 0 = t^3-t^2-9t-7$

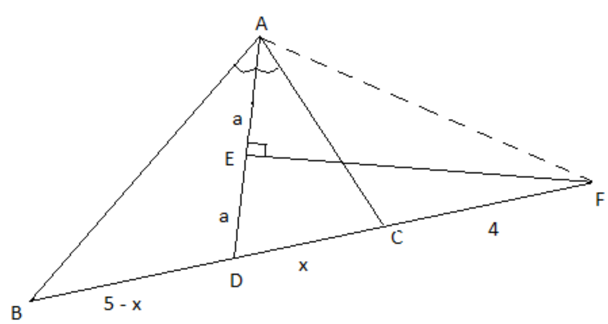
$= (t+1)\left(t^2-2t-7\right) \Rightarrow t=-1$ or $t = 1\pm 2\sqrt{2}$. Checking each case individually:

- $t=-1$ yields $f(1)=-1$, so that doesn't work.
- $t = 1-2\sqrt{2}$ yields $f\left(\sqrt{3-2\sqrt{2}}\right) = \frac{4-2\sqrt{2}}{-2\sqrt{2}} \Rightarrow f\left(\sqrt{2}-1\right) = 1-\sqrt{2}$, so that doesn't work.
- $t = 1+2\sqrt{2}$ yields $f\left(\sqrt{3+2\sqrt{2}}\right) = \frac{4+2\sqrt{2}}{2\sqrt{2}} \Rightarrow f\left(1+\sqrt{2}\right) = 1+\sqrt{2}$, so that does work. However,

the question asks which number gets fixed by f, so the answer is $1+\sqrt{2}$.

30. The picture to the left contains all information given in the problem. Draw line segment $\overline{AF}$, which also has length $4+x$ since $\Delta AEF \sim \Delta DEF$. By labeling all angles in the diagram, it is easy to show that $\Delta ACF \sim \Delta BAF$. Therefore, by similarity of triangles, $\frac{4}{4+x} = \frac{4+x}{9} \Rightarrow 4+x=6$.

Since $\left|\overline{FD}\right| = 4+x$, $\left|\overline{FD}\right| = 6$.



Tiebreakers

TB1. Start with the prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53,… . Of this partial list, 3, 5, 11, 17, 29, and 41 are all Chen primes because the number that is two larger is also prime. This reduces the list of potential non-Chen primes down to 2, 7, 13, 19, 23, 31, 37, 43, 47, 53. To see which of these has a number two larger that is semiprime, we will list such a factorization:

$n=2$: $4=2\cdot 2$
$n=7$: $9=3\cdot 3$
$n=13$: $15=5\cdot 3$
$n=19$: $21=7\cdot 3$
$n=23$: $25=5\cdot 5$
$n=31$: $33=11\cdot 3$
$n=37$: $39=13\cdot 3$

However, 43 is not semiprime because 45 has $3^2\cdot 5$ as its prime factorization, so it cannot be written as the product of two primes. Thus, 43 is the smallest prime that is not a Chen prime.

TB2. We'll again start with the prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29,…, and we'll begin the procedure:

$n=2$: 2, 4, 16, 37, 58, 89, 145, 42, 20, 4, …, and the cycle repeats without a 1
$n=3$: 3, 9, 81, 65, 61, 37, …, and the cycle repeats as in the previous case without a 1
$n=5$: 5, 25, 29, 85, 89, and the cycle repeats as in the first case without a 1
$n=7$: 7, 49, 97, 130, 10, 1, so 7 is the first happy prime
$n=11$: 11, 2, …, and the cycle repeats as in the first case without a 1
$n=13$: 13, 10, 1, so 13 is the second happy prime
$n=17$: 17, 50, 25, and the cycle repeats as in the third case without a 1
$n=19$: 19, 82, 68, 100, 1, so 19 is the third happy prime.

TB3. We'll again start with the prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, …, and we'll look at the condition of being a Higgs prime:

2 is the first Higgs prime
3 is also a Higgs prime because 2 divides 2^2
5 is also a Higgs prime because 4 divides $(2\cdot 3)^2$
7 is also a Higgs prime because 6 divides $(2\cdot 3\cdot 5)^2$
11 is also a Higgs prime because 10 divides $(2\cdot 3\cdot 5\cdot 7)^2$
13 is also a Higgs prime because 12 divides $(2\cdot 3\cdot 5\cdot 7\cdot 11)^2$

17 is not a Higgs prime because 16 does not divide $(2\cdot3\cdot5\cdot7\cdot11\cdot13)^2$, so 17 is the first prime that is not also a Higgs prime.