1. There are 50 grams after the first 5730 years and 25 grams after the second 5730 years.

2. The population doubles every two days. A million-fold growth is approximately 20 doublings ($2^{20} \approx 10^6$). Therefore, it takes 40 days to increase to a million-fold.

$$2^4 = 16$$
$$2^5 = 32$$
$$2^6 = 64$$
$$\vdots$$
$$2^{10} = 1024 \Rightarrow 2^{20} = (1024)^2$$

3. The units digit repeats after 4 iterations. When we divide 2013 by 4, the remainder is 1, so 3^{2013} has the same units digit as 3^1.

$$3^0 = 1$$
$$3^1 = 3$$
$$3^2 = 9$$
$$3^3 = 27$$
$$3^4 = 81$$
$$\vdots$$

4. $f(0) = a \cdot 2^0 = 5$ so $a = 5$.

$$f(3) = 5 \cdot 2^{3b} = 20\sqrt{2}$$
$$2^{3b} = 4\sqrt{2}$$
$$2^{3b} = \sqrt{32}$$
$$2^{3b} = \left(2^5\right)^{1/2}$$
$$3b = \frac{5}{2}$$
$$b = \frac{5}{6}$$

5. Since the rectangle lies in the first and second quadrants only, the width is 2*x*. The height is e^{-x^2} so the area is $A = 2x \cdot e^{-x^2}$

6. Using log rules,

$$\ln\frac{\left(x^2+3\right)^2}{x\cdot\sqrt[3]{x^2+1}} = \ln\left(x^2+3\right)^2 - \ln\left[x\cdot\left(x^2+1\right)^{1/3}\right]$$

$$= 2\ln\left(x^2+3\right) - \left[\ln x + \ln\left(x^2+1\right)^{1/3}\right]$$

$$= 2\ln\left(x^2+3\right) - \ln x - \frac{1}{3}\ln\left(x^2+1\right)$$

7. $\frac{1}{r} = \sqrt{\frac{1}{100}}$

$\frac{1}{r} = \frac{1}{10}$

$r = 10$

$r^3 = 1000$

8. Definition of an exponential function

9. $\log_2 x + \log_2(x+6) = 4$

$\log_2\left(x(x+6)\right) = 4$

$x(x+6) = 2^4$

$x^2 + 6x = 16$

10. Points on *f* are $(1,0),(2,1),(4,2)$ so the system of equations is $\begin{cases} a+b+c=0 \\ 4a+2b+c=1 \\ 16a+4b+c=2 \end{cases}$.

$f(x) = \frac{-1}{6}x^2 + \frac{3}{2}x - \frac{4}{3}$. $f(3) = \frac{5}{3}$

11. Using $P = P_0 e^{rt}$, set up the equation as

$$\begin{aligned} 7200 &= 1800e^{(0.04)t} \\ 4 &= e^{0.04t} \\ \ln 4 &= 0.04t \\ 2\ln 2 &= 0.04t \\ 2 \cdot 0.693 &\approx 0.04t \\ t &\approx 35\text{days} \end{aligned}$$

12. $4^1 \cdot 4^2 \cdot 4^3 \ldots 4^{100} = 4^{(1+2+3\ldots 100)} = 4^{5050}$. Therefore,

$$\log_{1024}\left(\prod_{j=1}^{100} 4^j\right) = \log_{1024}\left(4^{5050}\right) = 5050 \cdot \log_{1024} 4 = 5050 \cdot \log_{1024}\left(1024^{1/5}\right) = 5050 \cdot \frac{1}{5} = 1010$$

13. It should be $s(t) = 10(1.05)^t$

14. I. 0+1+16+81+...+100000000=1+16+81+...+100000000. True

II. $\sum_{j=0}^{100} 2 = 202 \neq 200$ False

III. 2+3+4+...+102 = 5252 ≠ 2 + (0+1+2+...+100)=5052 False

IV. 4+9+16+...+10201 ≠ 0+1+4+9801 False

V. 0+1+8+27+...+1000000 ≠ 5050^3 False

15. $2\log_3(x-2y) = \log_3 x + \log_3 y$

$$\begin{aligned} &\log_3 (x-2y)^2 = \log_3 xy \\ &(x-2y)^2 = xy \\ &x^2 - 5xy + 4y^2 = 0 \\ &(x-4y)(x-y) = 0 \\ &x = 4y \text{ or } x = y \\ &\frac{x}{y} = 4 \text{ or } \frac{x}{y} = 1 \end{aligned}$$

However, $\frac{x}{y} = 1$ is not allowed by $\log_3(x-2y)$

16. I. True. Let $x = b^y$. $f(x) = \log_b x$ so by substitution, $f(x) = \log_b b^y = y$

Let $x = (b)^y = (b^{-1})^{-y} = \left(\frac{1}{b}\right)^{-y}$. $g(y) = \log_{1/b} x$ so by substitution, $g(x) = \log_{1/b}\left(\frac{1}{b}\right)^{-y} = -y$.

II. True. $f(x) = \log_b x = y \Leftrightarrow x = b^y$. If $x = 1$ then $y = 0$.

$g(x) = \log_{1/b} x = y \Leftrightarrow \left(\frac{1}{b}\right)^y = x$. If $x = 1$ then $y = 0$.

III. False. $f(x) = \log_b x = y \Leftrightarrow x = b^y$. If $y = 1$ then $x = b$.

However, $g(x) = \log_{1/b} x = y \Leftrightarrow x = \left(\frac{1}{b}\right)^y = \left(b^{-1}\right)^y = b^{-y}$. If $y = 1$ then $x = -b$.

IV. True. See graph

V. True. f and g pass the horizontal line test.

17. $1568 = 2^5 \cdot 7^2$

$$M \log_{1568} 7 + A \log_{1568} 2 = T$$
$$\log_{1568}\left(7^M \cdot 2^A\right) = T$$
$$7^M \cdot 2^A = 1568^T$$
$$7^M \cdot 2^A = \left(7^2 \cdot 2^5\right)^T$$
$$7^M \cdot 2^A = 7^{2T} \cdot 2^{5T}$$
$$M = 2T$$
$$A = 5T$$

Since M, A, T have no common factors greater than 1, M=2, A=5, T=1.

18. $\log_{2^{(2n)}} x = \dfrac{\log_2 x}{\log_2 2^{2n}} = \dfrac{\log_2 x}{2^n} = \dfrac{1}{2^n} \cdot \log_2 x = \left(\dfrac{1}{2}\right)^n \cdot \log_2 x$

$$8 = \left[\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n\right] \cdot \log_2 x$$
$$8 = \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right] \cdot \log_2 x$$
$$8 = \log_2 x$$
$$2^8 = x$$
$$x = 256$$

19. Write out some terms to see the pattern.

$$\sum_{n=1}^{4095} \log_4\left(\frac{n+1}{n}\right) = \log_4 \frac{2}{1} + \log_4 \frac{3}{2} + \log_4 \frac{4}{3} + \ldots + \log_4 \frac{4095}{4094} + \log_4 \frac{4096}{4095}$$
$$= \log_4\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{4095}{4094} \cdot \frac{4096}{4095}\right)$$
$$= \log_4 4096 = \log_4 4^6 = 6$$

20. $\log_3\left(10 - 3^x\right) = 2 - x$

$$10 - 3^x = 3^{2-x}$$
$$10 - 3^x = \frac{9}{3^x}$$

let $y = 3^x$ then $y(10 - y) = 9 \Rightarrow y^2 - 10y + 9 = 0 \Rightarrow y = 9$ or $y = 1$. So $x = 2$ or $x = 0$

21. By rules of graph translations.

22. $\displaystyle\sum_{n=0}^{10} 2^n = 1 + 2 + 4 + \ldots + 1024 = 1 + 2 \cdot \left(\frac{1 - 2^{10}}{1 - 2}\right) = 2047$

23. $\left(5^x + 5^{-x}\right)^2 = 5^{2x} + 2 + 5^{-2x} = 25^x + 25^{-x} + 2 = 47 + 2 = 49$ Therefore $5^x + 5^{-x} = 7$

24. $\log_2 18 = \log_2 3^2 \cdot 2 = 2\log_2 3 + 1 = 2 \cdot \dfrac{\log_8 3}{\log_8 2} + 1 = 2 \cdot \dfrac{\log_8 3}{1/3} + 1 = 6 \cdot \log_8 3 + 1 = 6k + 1$

25. $2^{x-2} = 225$

$2^x \cdot 2^{-2} = 3^2 \cdot 5^2 = 3^2 \cdot \left(\frac{10}{2}\right)^2$

$2^x = 3^2 \cdot \left(\frac{10}{2}\right)^2 \cdot 2^2$

$x\log 2 = 2\log 3 + 2(\log 10 - \log 2) + 2\log 2 = 2\log 3 + 2$

$x = \frac{2\log 3 + 2}{\log 2} = \frac{2 \cdot .4771}{.3010} = 9.81462$

26. $2^{x+7} - 3(2^x) = 5^{y+2} - 9(5^y)$

$2^x \cdot 2^7 - 3 \cdot 2^x = 5^y \cdot 5^2 - 9 \cdot 5^y$

$2^x(2^7 - 3) = 5^y(5^2 - 9)$

$2^x(125) = 5^y(16) \Rightarrow 2^x \cdot 5^3 = 5^y \cdot 2^4 \Rightarrow y = 3$ and $x = 4$

27. $3(4x^2 - 9) = (4x^2 - 9)e^{9-x}$

$3(4x^2 - 9) - (4x^2 - 9)e^{9-x} = 0$

$(4x^2 - 9)(3 - e^{9-x}) = 0$

$x = \pm\frac{3}{2}$ or $x = 9 - \ln 3$

28. $f(g(x)) = 4^{\log_2\left(2^{\log_2 5}\right)} = 4^{\log_2 5} = 2^{2 \cdot \log_2 5} = 5^2 = 25$

29. $f(x) = x\sqrt{x\sqrt{x\sqrt{\cdots}}}$

$f(x) = x\sqrt{f(x)} \quad \Rightarrow \quad x = \sqrt{f(x)} \quad \Rightarrow f(x) = x^2$

so $\frac{\log f(x)}{\log x} = \frac{\log x^2}{\log x} = \frac{2\log x}{\log x} = 2$

30. $S_7 = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6$ by definition of a geometric series.

We know $a + ar = 20 \Rightarrow (1+r) = \dfrac{20}{a}$ and $ar^5 + ar^6 = -\dfrac{5}{8} \Rightarrow ar^5(1+r) = -\dfrac{5}{8}$. By substitution,

$ar^5\left(\dfrac{20}{a}\right) = -\dfrac{5}{8} \Rightarrow r^5 = -\dfrac{5}{160} \Rightarrow r = -\dfrac{1}{2}$ and $a = 40$. Now we can use the formula for a

geometric series. $\dfrac{40 - 40\cdot\left(-\dfrac{1}{2}\right)^7}{1 - \left(-\dfrac{1}{2}\right)} = \dfrac{215}{8}$

Bonus: $\dfrac{x}{3} = x^2 - x^3 + x^4 - \cdots$ If we divide both sides of the equation by x^2 we get

$$\frac{1}{3x} = 1 - x + x^2 - \cdots$$
$$= \sum_{n=0}^{\infty}\left(-x^n\right) = \frac{1}{1-(-x)} = \frac{1}{x+1}$$

so $\dfrac{1}{3x} = \dfrac{1}{x+1} \Rightarrow 3x = x + 1 \Rightarrow x = \dfrac{1}{2}$. $x = 0$ is the trivial case.